Z-Transform

The Laplace transform deals with continuous functions and can be used to solve differential equations.

Similarly, the **Z-transform** deals with discrete sequences and the recurrence relations - or difference equations.

Sequences

Consider the sequence ..., $3^{-2}, 3^{-1}, 3^0, 3^1, 3^2$.... It has a general form 3^k and using a shorthand notation we can write the sequence as $\left\{3^k\right\}_{-\infty}^{\infty}$ indicating also that the powers range from $-\infty$ to ∞.

The sum

$$\sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k = \ldots\left(\frac{3}{z}\right)^{-1}+\left(\frac{3}{z}\right)^0+\left(\frac{3}{z}\right)^1+\left(\frac{3}{z}\right)^2\ldots \tag{1}$$

is called the **Z-transform** of the sequence, $\mathcal{Z}\left\{3^k\right\}_{-\infty}^{\infty}$ and is denoted $F(z)$ where the complex number z is chosen to ensure that the sum is finite.

We say that $\left\{3^k\right\}_{-\infty}^{\infty}$ and $\mathcal{Z}\left\{3^k\right\}_{-\infty}^{\infty} = F(z) = \sum_{k=-\infty}^{\infty}\left(\frac{3}{z}\right)^k$ form a **Z-transform pair**.

For our purposes we shall consider only **causal sequences** of the form $\{x_k\}_0^\infty$ where $x_k = 0$ for $k < 0$:

$$\mathcal{Z}\{x_k\} = F(z) = \sum_{k=0}^{\infty} \frac{x_k}{z^k} \tag{2}$$

Example 1: $\{\delta_k\} = \{1, 0, 0, \ldots\}$

$$\mathcal{Z}\{\delta_k\} = F(z) = 1 + \frac{0}{z} + \frac{0}{z^2} + \frac{0}{z^3} + \ldots = 1$$

Example 2: The **unit step sequence**: $\{u_k\} = \{1, 1, 1, \ldots\}$

$$\mathcal{Z}\{u_k\} = F(z) = \sum_{k=0}^{\infty} \frac{1}{z^k} = 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots$$

Comparing this to the series expansion of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ we get

$$F(z) = \frac{1}{1 - \frac{1}{z}} \quad \text{provided} \quad \left|\frac{1}{z}\right| < 1$$

or

$$F(z) = \frac{z}{z - 1} \quad \text{provided} \quad |z| > 1$$

Example 3: $\{x_k\} = \{1, a, a^2, a^3, \ldots\} = \{a^k\}$

$$\mathcal{Z}\left\{a^k\right\} = \sum_{k=0}^{\infty} \frac{a^k}{z^k} = 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ldots$$

Comparing this to the series expansion of $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ which is valid for $|x| < 1$ then

$$\begin{aligned} F(z) &= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \ldots \\ &= \frac{1}{1 - \frac{a}{z}} \quad \text{provided} \quad \left|\frac{a}{z}\right| < 1 \end{aligned}$$

or

$$F(z) = \frac{z}{z - a} \quad \text{provided} \quad |z| > |a|$$

Example 4: $\{x_k\} = \{0, 1, 2, 3, 4, \ldots\} = \{k\}$

$$\mathcal{Z}\{k\} = F(z) = \sum_{k=0}^{\infty} \frac{k}{z^k} = 0 + \frac{1}{z} + \frac{2}{z^2} + \frac{3}{z^3} + \frac{4}{z^4} +$$

Comparing this with the derivative of $(1 - x)^{-1}$ and its series expansion

$$1 + 2x + 3x^2 + 4x^3 + \ldots = \frac{\mathrm{d}}{\mathrm{d}x}(1 + x + x^2 + x^3 + x^4 + \ldots) = \frac{\mathrm{d}}{\mathrm{d}x}(1 - x)^{-1} = \frac{1}{(1 - x)^2}$$

we can write

$$zF(z) = 1 + \frac{2}{z} + \frac{3}{z^2} + \frac{4}{z^3} + \ldots = \frac{1}{(1 - 1/z)^2}$$

and dividing both sides by z we obtain

$$F(z) = \frac{1}{z(1 - 1/z)^2} = \frac{z}{(z - 1)^2}$$

Table of Z transforms

Sequence	Transform $F(z)$	Permitted values of z
$\{\delta_k\} = \{1, 0, 0, 0, \ldots\}$	1	All values of z
$\{u_k\} = \{1, 1, 1, 1, \ldots\}$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
$\{k\} = \{0, 1, 2, 3, \ldots\}$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
$\{k^2\} = \{0, 1, 4, 9, \ldots\}$	$\frac{z(z+1)}{(z-1)^3}$	$\lvert z\rvert > 1$
$\{k^3\} = \{0, 1, 8, 27, \ldots\}$	$\frac{z(z^2+4z+1)}{(z-1)^4}$	$\lvert z\rvert > 1$
$\{a^k\} = \{1, a, a^2, a^3, \ldots\}$	$\frac{z}{(z-a)}$	$\lvert z\rvert > \lvert a\rvert$
$\{ka^k\} = \{0, a, 2a^2, 3a^3, \ldots\}$	$\frac{az}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$

Properties of Z transforms

1 Linearity

The Z transform is a linear transform, that is

$$\mathcal{Z}\left\{ax_k + by_k\right\} = a\mathcal{Z}\left\{x_k\right\} + b\mathcal{Z}\left\{y_k\right\} \tag{3}$$

where a and b are constants.

Example 5: $3\{k\} - 5\left\{e^{-2k}\right\}$

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2}$$

$$\mathcal{Z}\left\{a^k\right\} = \frac{z}{z-a} \quad \Rightarrow \quad \mathcal{Z}\left\{e^{-2k}\right\} = \frac{z}{z-e^{-2}}$$

Consequently

$$\begin{aligned} \mathcal{Z}\left(3\{k\} - 5\left\{e^{-2k}\right\}\right) &= \frac{3z}{(z-1)^2} - \frac{5z}{(z-e^{-2})} \\ &= \frac{-5z^3 + 13z^2 - z(3e^{-2} + 5)}{(z-1)^2(z-e^{-2})} \end{aligned}$$

2 First shifting theorem

If $\mathcal{Z}\{x_k\} = F(z)$ then

$$\mathcal{Z}\{x_{k+m}\} = z^m F(z) - [z^m x_0 + z^{m-1} x_1 + \ldots + z x_{m-1}] \quad (4)$$

is the Z transform of the sequence that has been shifted by m places to the left.

Example:

$$\begin{aligned} \mathcal{Z}\{x_{k+1}\} &= zF(z) - zx_0 \\ \mathcal{Z}\{x_{k+2}\} &= z^2 F(z) - z^2 x_0 - zx_1 \end{aligned}$$

Example 6: $\left\{4^{k+3}\right\}$

Given

$$\mathcal{Z}\left\{4^k\right\} = \frac{z}{z-4}$$

$$\begin{aligned}\mathcal{Z}\left\{4^{k+3}\right\} &= z^3\mathcal{Z}\left\{4^k\right\} - [z^3 4^0 + z^2 4^1 + z4^2] \\ &= z^3\frac{z}{z-4} - [z^3 + 4z^2 + 16z] = \frac{z^4}{z-4} - [z^3 + 4z^2 + 16z] \\ &= \frac{z^4 - [z^3 + 4z^2 + 16z](z-4)}{z-4} = \frac{z^4 - (z^4 - 64z)}{z-4} \\ &= \frac{64z}{z-4}\end{aligned}$$

We have just derived the Z transform of the sequence $\{64, 256, 1024, ...\}$ by shifting $\{1, 4, 16, 64, 256, ...\}$ three places to the left and loosing the first three terms.

Example 7: $\{k+1\}$

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2}$$

$$\mathcal{Z}\{k+1\} = z\frac{z}{(z-1)^2} - [z \times 0]$$

$$= \frac{z^2}{(z-1)^2}$$

3 Second shift theorem

If $\mathcal{Z}\{x_k\} = F(z)$ then

$$\mathcal{Z}\{x_{k-m}\} = z^{-m}F(z) \tag{5}$$

is the Z transform of the sequence that has been shifted by m places to the right.

Example 8:
Given

$$\mathcal{Z}\{x_k\} = \frac{z}{z-1}$$

then

$$\mathcal{Z}\{x_{k-3}\} = z^{-3}\frac{z}{z-1} = \frac{1}{z^2(z-1)}$$

We thus derived the Z transform of the sequence $\{0, 0, 0, 1, 1, 1, ...\}$ by shifting $\{1, 1, 1, ...\}$ three places to the right and defining the first three terms as zeros.

Example 9:

Given the Z transform

$$\mathcal{Z}\{x_k\} = \frac{1}{z-a}$$

where a is a constant. The sequence $\{x_k\}$ is

$$\left\{a^{k-1}\right\}$$

because

$$\frac{1}{z-a} = \frac{1}{z} \times \frac{z}{z-a} = z^{-1}F(z)$$

where $F(z) = \mathcal{Z}\left\{a^k\right\}$ and so

$$\frac{1}{z-a} = \mathcal{Z}\left\{a^{k-1}\right\}$$

4 Translation

If the sequence $\{x_k\}$ has the Z transform $\mathcal{Z}\{x_k\} = F(z)$ then the sequence $\left\{a^k x_k\right\}$ has the Z transform $\mathcal{Z}\left\{a^k x_k\right\} = F(a^{-1}z)$.

Example 10:

$$\mathcal{Z}\{k\} = \frac{z}{(z-1)^2}$$

so

$$\mathcal{Z}\left\{2^k k\right\} = F\left(2^{-1}z\right) = \frac{2^{-1}z}{\left(2^{-1}z-1\right)^2} = \frac{2z}{(z-2)^2}$$

5 Final value theorem

For the sequence $\{x_k\}$ with Z transform $F(z)$

$$\lim_{k\to\infty} x_k = \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} \tag{6}$$

provided that $\lim_{k\to\infty} x_k$ exists.

Example: The sequence $\left\{\left(\frac{1}{2}\right)^k\right\}$ has the Z transform

$$F(z) = \frac{z}{z-\frac{1}{2}} = \frac{2z}{2z-1}$$

so

$$\lim_{k\to\infty}\left\{\left(\frac{1}{2}\right)^k\right\} = \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} = \lim_{z\to 1}\left\{\frac{2(z-1)}{2z-1}\right\} = 0$$

Example 11:

Using the final value theorem, the final value of the sequence with the Z transform

$$F(z) = \frac{10z^2 + 2z}{(z-1)(5z-1)^2} \tag{7}$$

is calculated as follows

$$\begin{aligned}
\lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)F(z)\right\} &= \lim_{z\to 1}\left\{\left(\frac{z-1}{z}\right)\frac{10z^2+2z}{(z-1)(5z-1)^2}\right\} \\
&= \lim_{z\to 1}\left\{\frac{10z+2}{(5z-1)^2}\right\} \\
&= \frac{12}{16} \\
&= 0.75
\end{aligned}$$

6 The initial value theorem

For the sequence $\{x_k\}$ with the Z transform $F(z)$

$$x_0 = \lim_{z\to\infty} \{F(z)\} \tag{8}$$

Example:

The sequence $\left\{a^k\right\}$ has the Z transform $F(z) = \frac{z}{z-a}$ and, using the l'Hospital rule,

$$\lim_{z\to\infty} F(z) = \lim_{z\to\infty} \frac{z}{z-a} = \lim_{z\to\infty} \frac{1}{1} = 1$$

Furthermore $x_0 = a^0 = 1$.

7 The derivative of the transform

If $\mathcal{Z}\{x_k\} = F(z)$ then

$$-zF'(z) = \mathcal{Z}\{kx_k\} \tag{9}$$

Proof:

$$F(z) = \sum_{k=0}^{\infty} x_k z^{-k}$$

and so

$$\begin{aligned} F'(z) &= \sum_{k=0}^{\infty} x_k(-k)z^{-k-1} = -\frac{1}{z}\sum_{k=0}^{\infty} x_k k z^{-k} \\ &= -\frac{1}{z}\mathcal{Z}\{kx_k\} \end{aligned}$$

Example 12:

The sequence $\left\{a^k\right\}$ has the Z transform $F(z) = \frac{z}{z-a}$ and so the sequence $\left\{ka^k\right\}$ has Z transform

$$\mathcal{Z}\left\{kx_k\right\} = -zF'(z) = -z\left(\frac{z}{z-a}\right)' = -z\left(\frac{z-a-z}{(z-a)^2}\right) = \frac{az}{(z-a)^2}$$

Notice that this is in agreement with the table of transforms.

Inverse Z transforms

If the sequence $\{x_k\}$ has Z transform $\mathcal{Z}\{x_k\} = F(z)$, the inverse transform is defined as

$$\mathcal{Z}^{-1}F(z) = \{x_k\}$$

To carry out the inverse Z transform, we will usually need to perform some manipulation, the most often using the partial fraction decomposition.

Example 13:

The sequence $\{x_k\}$ has Z transform $F(z) = \frac{z}{z^2-5z+6}$. We first perform the partial fraction decomposition

$$F(z) = \frac{z}{z^2 - 5z + 6} = \frac{z}{(z-2)(z-3)} = \frac{A}{z-2} + \frac{B}{z-3} = \frac{A(z-3) + B(z-2)}{(z-2)(z-3)}$$

Equating numerators and solving for A and B gives $A = -2$ and $B = 3$. So

$$F(z) = \frac{3}{z-3} - \frac{2}{z-2}$$

The nearest Z transform in the table to either of these two partial fractions is $\mathcal{Z}\left\{a^k\right\} = \frac{z}{z-a}$ so we write

$$\begin{aligned} F(z) &= \frac{3}{z-3} - \frac{2}{z-2} = \frac{3}{z} \times \frac{z}{z-3} - \frac{2}{z} \times \frac{z}{z-2} \\ &= 3 \times z^{-1} \mathcal{Z}\left\{3^k\right\} - 2 \times z^{-1} \mathcal{Z}\left\{2^k\right\} \end{aligned}$$

The inverse Z transform is then

$$\begin{aligned} \mathcal{Z}^{-1} F(z) &= 3 \times \left\{3^{k-1}\right\} - 2 \times \left\{2^{k-1}\right\} \\ &= \left\{3^k\right\} - \left\{2^k\right\} \\ &= \left\{3^k - 2^k\right\} \end{aligned}$$

giving $x_k = 3^k - 2^k$.

We can solve this problem also without using the second shift theorem. We consider instead the partial fraction decomposition of $\frac{F(z)}{z}$:

$$\begin{aligned}\frac{F(z)}{z} &= \frac{1}{z} \times \frac{z}{z^2 - 5z + 6} = \frac{1}{z^2 - 5z + 6} = \frac{1}{(z-2)(z-3)} \\ &= \frac{A}{z-2} + \frac{B}{z-3} = \frac{A(z-3) + B(z-2)}{(z-2)(z-3)}\end{aligned}$$

Equating numerators and solving for A and B yields $A = -1$ and $B = 1$, so that

$$\frac{F(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2} \qquad \Rightarrow \qquad F(z) = \frac{z}{z-3} - \frac{z}{z-2} = \mathcal{Z}\left\{3^k\right\} - \mathcal{Z}\left\{2^k\right\}$$

The final result is

$$\begin{aligned}\mathcal{Z}^{-1}F(z) &= \left\{3^k\right\} - \left\{2^k\right\} \\ &= \left\{3^k - 2^k\right\}\end{aligned}$$

Example 14:

The sequence $\{x_k\}$ has Z transform

$$F(z) = \frac{5z}{(z^2 - 4z + 4)(z + 2)}$$

We divide $F(z)$ by z and perform the partial fraction decomposition

$$\begin{aligned}
\frac{F(z)}{z} &= \frac{1}{z} \times \frac{5z}{(z^2 - 4z + 4)(z + 2)} = \frac{5}{(z - 2)^2(z + 2)} \\
&= \frac{A}{(z - 2)^2} + \frac{B}{z - 2} + \frac{C}{z + 2} \\
&= \frac{A(z + 2) + B(z - 2)(z + 2) + C(z - 2)^2}{(z - 2)^2(z + 2)}
\end{aligned}$$

Equating numerators and solving for A, B and C yields $A = 5/4$, $B = -5/16$ and $C = 5/16$, so

$$\frac{F(z)}{z} = \frac{5/4}{(z-2)^2} - \frac{5/16}{z-2} + \frac{5/16}{z+2}$$

giving

$$F(z) = \frac{5}{8} \times \frac{2z}{(z-2)^2} - \frac{5}{16} \times \frac{z}{z-2} + \frac{5}{16} \times \frac{z}{z+2}$$

The inverse Z transform is then

$$\begin{aligned} \mathcal{Z}^{-1}F(z) &= \frac{5}{8} \times \left\{k2^k\right\} - \frac{5}{16} \times \left\{2^k\right\} + \frac{5}{16} \times \left\{(-2)^k\right\} \\ &= \left\{\frac{5}{16}\left[(2k-1)2^k + (-2)^k\right]\right\} \end{aligned}$$