Fourier integral

Fourier series were used to represent a function f defined of a *finite* interval $(-p, p)$ or $(0, L)$. It converged to f and to its periodic extension. In this sense Fourier series is associated with *periodic* functions.

Fourier integral represents a certain type of *nonperiodic* functions that are defined on either $(-\infty, \infty)$ or $(0, \infty)$.

From Fourier series to Fourier integral

Let a function f be defined on $(-p, p)$. The Fourier series of the function is then

$$\begin{aligned} f(x) &= \frac{1}{2p}\int_{-p}^{p} f(t)\,dt + \\ &+ \frac{1}{p}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\ \cos\frac{n\pi}{p}t\,dt\right)\cos\frac{n\pi}{p}x + \left(\int_{-p}^{p} f(t)\ \sin\frac{n\pi}{p}t\,dt\right)\sin\frac{n\pi}{p}x\right] \end{aligned} \tag{1}$$

If we let $\alpha_n = n\pi/p$, $\Delta\alpha = \alpha_{n+1} - \alpha_n = \pi/p$, we get

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\left(\int_{-p}^{p} f(t)\,dt\right)\Delta\alpha + \\ &+ \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\int_{-p}^{p} f(t)\ \cos\alpha_n t\,dt\right)\cos\alpha_n x + \left(\int_{-p}^{p} f(t)\ \sin\alpha_n t\,dt\right)\sin\alpha_n x\right]\Delta\alpha \end{aligned} \tag{2}$$

We now expand the interval $(-p, p)$ by taking $p \to \infty$ which implies that $\Delta\alpha \to 0$. Consequently,

$$\lim_{\Delta\alpha \to 0} \sum_{n=1}^{\infty} F(\alpha_n)\, \Delta\alpha \to \int_0^{\infty} F(\alpha)\, d\alpha$$

Thus, the limit of the first term in the Fourier series $\int_{-p}^{p} f(t)\, dt$ vanishes, and the limit of the sum becomes

$$f(x) = \frac{1}{\pi} \int_0^{\infty} \left[\left(\int_{-\infty}^{\infty} f(t)\ \cos\alpha t\ dt \right) \cos\alpha x + \left(\int_{-\infty}^{\infty} f(t)\ \sin\alpha t\ dt \right) \sin\alpha x \right] d\alpha$$

This is the **Fourier** integral of f on the interval $(-\infty, \infty)$.

Definition: Fourier integral

The Fourier integral of a function f defined on the interval $(-\infty, \infty)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty [A(\alpha)\ \cos\alpha x + B(\alpha)\ \sin\alpha x]\, d\alpha \tag{3}$$

where

$$A(\alpha) = \int_{-\infty}^{\infty} f(x)\ \cos\alpha x\, dx \tag{4}$$

$$B(\alpha) = \int_{-\infty}^{\infty} f(x)\ \sin\alpha x\, dx \tag{5}$$

Convergence of a Fourier integral

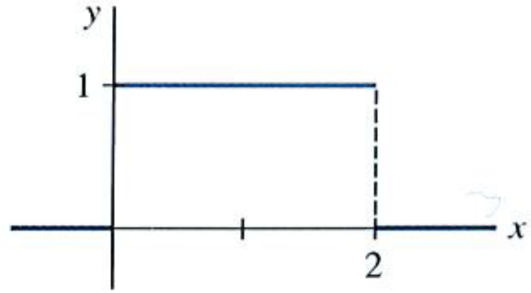
Theorem: Conditions for convergence

Let f and f' be piecewise continuous on every finite interval, and let f be absolutely integrable on $(-\infty, \infty)$ (i.e. the integral $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges). Then the Fourier integral of f on the interval converges for $f(x)$ at a point of continuity. At a point of dicontinuity, the Fourier integral will converge to the average

$$\frac{f(x+) + f(x-)}{2}$$

where $f(x+)$ and $f(x-)$ denote the limit of f at x from the right and from the left, respectively.

Example 1: Fourier integral representation



$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 < x < 2 \\ 0, & x > 2 \end{cases}$$

The function satisfies the assumptions of the theorem above, so the Fourier integral can be computed as follows:

$$\begin{aligned}
A(\alpha) &= \int_{-\infty}^{\infty} f(x)\ \cos\alpha x\, dx \\
&= \int_{-\infty}^{0} f(x)\ \cos\alpha x\, dx + \int_{0}^{2} f(x)\ \cos\alpha x\, dx + \int_{2}^{\infty} f(x)\ \cos\alpha x\, dx \\
&= \int_{0}^{2} \cos\ \alpha x\, dx = \frac{\sin 2\alpha}{\alpha} \\
B(\alpha) &= \int_{-\infty}^{\infty} f(x)\ \sin\alpha x\, dx = \int_{0}^{2} \sin\ \alpha x\, dx = \frac{1 - \cos 2\alpha}{\alpha}
\end{aligned}$$

Substituting these coefficients into the Fourier integral

$$f(x) = \frac{1}{\pi}\int_0^\infty \left[\left(\frac{\sin\ 2\alpha}{\alpha}\right)\cos\ \alpha x + \left(\frac{1-\cos\ 2\alpha}{\alpha}\right)\sin\ \alpha x\right] d\alpha$$

Using trigonometric identities the last integral simplifies to

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\ \alpha\ \cos\ \alpha(x-1)}{\alpha} d\alpha$$

Comment: The Fourier integral can be used to evaluate integrals. For example, at $x = 1$, the result above converges to $f(1)$; that is

$$\int_0^\infty \frac{\sin\alpha}{\alpha}\, d\alpha = \frac{\pi}{2}$$

The integrant $(\sin x)/x$ does not posses antiderivative that is an elementary function.

Cosine and sine integrals

Definition: Fourier cosine and sine integrals

(i) The Fourier integral of an even function on the interval $(-\infty, \infty)$ is the **cosine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty A(\alpha)\ \cos\alpha x\, d\alpha \tag{6}$$

where

$$A(\alpha) = \int_0^\infty f(x)\ \cos\alpha x\, dx \tag{7}$$

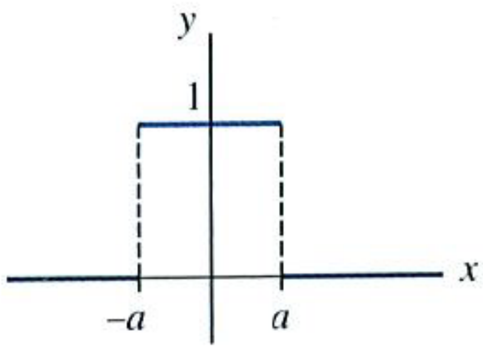
(ii) The Fourier integral of an odd function on the interval $(-\infty, \infty)$ is the **sine integral**

$$f(x) = \frac{2}{\pi}\int_0^\infty B(\alpha)\ \sin\alpha x\, d\alpha \tag{8}$$

where

$$B(\alpha) = \int_0^\infty f(x)\ \sin\alpha x\, dx \tag{9}$$

Example 2: Cosine integral representation



$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

This function is even, hence we can represent f by the Fourier cosine integral. We get

$$\begin{aligned} A(\alpha) &= \int_0^\infty f(x)\ \cos\alpha x\ dx = \int_0^a f(x)\ \cos\alpha x\ dx + \int_a^\infty f(x)\ \cos\alpha x\ dx \\ &= \int_0^a \cos\alpha x\ dx = \frac{\sin a\alpha}{\alpha} \end{aligned}$$

and so

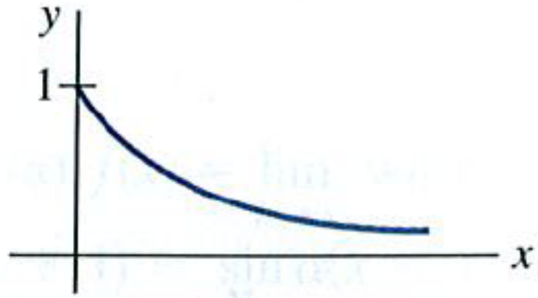
$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\alpha\ \cos\alpha x}{\alpha} d\alpha$$

The Fourier cosine and sine integrals, (6) and (8) respectively, can be used when f is neither odd not even and defined only on the half-line $(0, \infty)$.

In this case, (6) represents f on the interval $(0, \infty)$ and its even, but not periodic, extension to $(-\infty, 0)$.

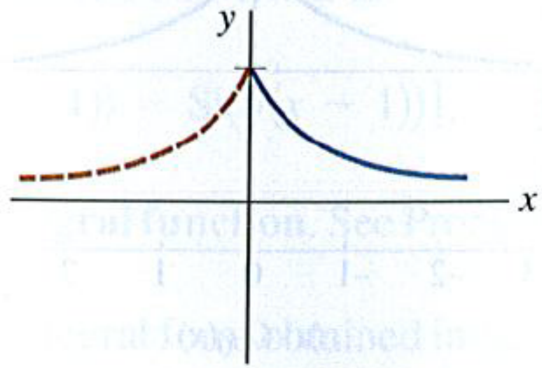
Similarly, (8) represents f on the interval $(0, \infty)$ and its odd, but not periodic, extension to $(-\infty, 0)$.

Example 3: Cosine and sine integral representations



$f(x) = e^{-x}$, $x > 0$

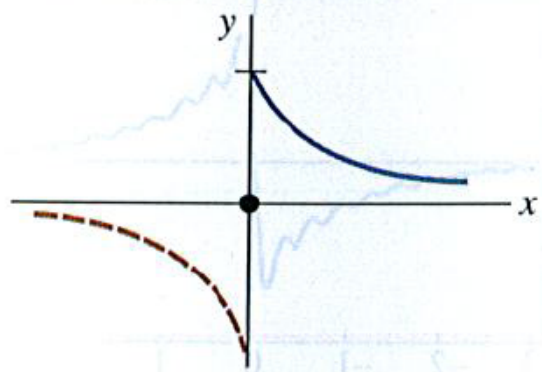
(a) A cosine integral:



$$
\begin{aligned}
A(\alpha) &= \int_0^\infty e^{-x} \cos \alpha x \, dx = \frac{1}{1+\alpha^2} \\
f(x) &= \frac{2}{\pi} \int_0^\infty \frac{\cos \alpha x}{1+\alpha^2} \, d\alpha
\end{aligned}
$$

(b) A sine integral:

$$
\begin{aligned}
B(\alpha) &= \int_0^\infty e^{-x} \sin \alpha x \, dx = \frac{\alpha}{1+\alpha^2} \\
f(x) &= \frac{2}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{1+\alpha^2} \, d\alpha
\end{aligned}
$$



Complex form

The Fourier integral (3) also possesses an equivalent **complex form**, or **exponential form**:

$$\begin{aligned}
f(x) &= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)[\cos\alpha t\cos\alpha x + \sin\alpha t\sin\alpha x]\,dt\,d\alpha \\
&= \frac{1}{\pi}\int_0^\infty \int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dt\,d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\cos\alpha(t-x)\,dt\,d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)[\cos\alpha(t-x) - i\sin\alpha(t-x)]\,dt\,d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \int_{-\infty}^\infty f(t)\,e^{-i\alpha(t-x)}\,dt\,d\alpha \\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \left(\int_{-\infty}^\infty f(t)\,e^{-i\alpha t}\,dt\right)e^{i\alpha x}\,d\alpha
\end{aligned}$$

In order to derive the complex form, we used a few observations and tricks:
(i) to get to the third line we used the fact that the integrand on the second line is an even function of α.
(ii) to get from the third to fourth line, we added to the integrand zero in the form of an integral of an odd function

$$i \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t) \ \sin \alpha(t - x) \, dt \, d\alpha = 0$$

The complex Fourier integral can be expressed as

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} C(\alpha) \, e^{i\alpha x} \, d\alpha \tag{10}$$

where

$$C(\alpha) = \int_{-\infty}^{\infty} f(x) \, e^{-i\alpha x} \, dx \tag{11}$$

The convergence of a Fourier integral can be examined in a manner that is similar to graphing partial sums of a Fourier series.

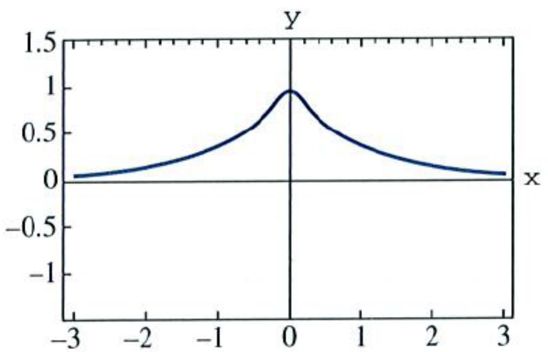
Example: By definition of an improper integral, the Fourier cosine integral representation of $f(x) = e^{-x}$, $x > 0$, can be written as $f(x) = \lim_{b\to\infty} F_b(x)$ where

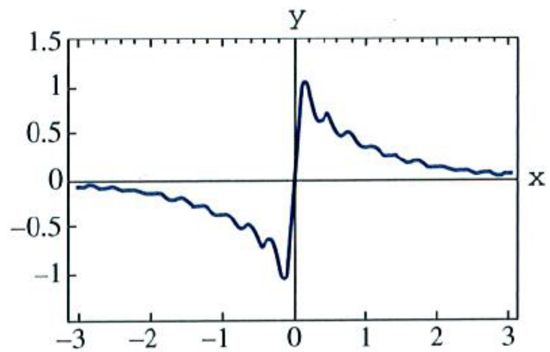
$$F_b(x) = \frac{2}{\pi}\int_0^b \frac{\cos\alpha x}{1+\alpha^2}d\alpha$$

and x is treated as a parameter.

Similarly, the Fourier sine representation of $f(x) = e^{-x}$, $x > 0$, can be written as $f(x) = \lim_{b\to\infty} G_b(x)$ where

$$G_b(x) = \frac{2}{\pi}\int_0^b \frac{\alpha\sin\alpha x}{1+\alpha^2}d\alpha$$





Fourier transform

We will now

- introduce a new integral transforms called **Fourier transforms**;
- expand on the concept of transform pair: an integral transform and its inverse;
- see that the inverse of an integral transform is itself another integral transform.

Transform pairs

Integral transforms appear in **transform pairs**: if $f(x)$ is transformed into $F(\alpha)$ by an integral transform

$$F(\alpha) = \int_a^b f(x)\, K(\alpha, x)\, dx$$

then the function$f$ can be recovered by another integral transform

$$f(x) = \int_a^b F(\alpha)\, H(\alpha, x)\, dx$$

called the **inverse transform**. The functions K and H in the integrands above are called the **kernels** of their respective transforms. For example $K(s, t) = e^{-st}$ is the kernel of the Laplace transform.

Fourier transform pairs

The Fourier integral is the source of three new integral transforms.

Definition: Fourier transform pairs

(i)
Fourier transform:

$$\mathcal{F}\{f(x)\} = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx = F(\alpha) \tag{12}$$

Inverse Fourier transform:

$$\mathcal{F}^{-1}\{F(\alpha)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha)\, e^{i\alpha x}\, d\alpha = f(x) \tag{13}$$

(ii)

Fourier sine transform:

$$\mathcal{F}_s\{f(x)\} = \int_0^\infty f(x)\ \sin \alpha x \, dx = F(\alpha) \tag{14}$$

Inverse Fourier sine transform:

$$\mathcal{F}_s^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\ \sin \alpha x \, d\alpha = f(x) \tag{15}$$

(iii)
Fourier cosine transform:

$$\mathcal{F}_c\{f(x)\} = \int_0^\infty f(x)\ \cos\alpha x\, dx = F(\alpha) \qquad (16)$$

Inverse Fourier cosine transform:

$$\mathcal{F}_c^{-1}\{F(\alpha)\} = \frac{2}{\pi}\int_0^\infty F(\alpha)\ \cos\alpha x\, d\alpha = f(x) \qquad (17)$$

Existence

The existence conditions for the Fourier transform are more stringent than those for the Laplace transform. For example, $\mathcal{F}\{1\}$, $\mathcal{F}_s\{1\}$ and $\mathcal{F}_c\{1\}$ do not exist.

Sufficient conditions for existence are that f be absolutely integrable on the appropriate interval and that f and f' are piecewise continuous on every finite interval.

Operational properties

Transforms of derivatives.

(i) Fourier transform

Supose that f is continuous and absolutely integrable on the interval $(-\infty, \infty)$ and f' is piecewise continuous on every finite interval. If $f(x) \to 0$ as $x \to \pm\infty$, then integration by parts gives

$$\begin{aligned}
\mathcal{F}\left\{f'(x)\right\} &= \int_{-\infty}^{\infty} f'(x)\, e^{-i\alpha x}\, dx = \left[f(x)\, e^{-i\alpha x}\right]_{-\infty}^{\infty} + i\alpha \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\
&= i\alpha \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \\
\text{That is: } \mathcal{F}\left\{f'(x)\right\} &= i\alpha F(\alpha) \qquad (18)
\end{aligned}$$

$$\mathcal{F}\left\{f'(x)\right\} = i\alpha F(\alpha)$$

Similarly, under the added assumptions that f' is continuous on $(-\infty, \infty)$, $f''(x)$ is piecewise continuous on every finite interval, and $f'(x) \to 0$ as $x \to \pm\infty$, we have

$$\mathcal{F}\left\{f''(x)\right\} = (i\alpha)^2 F(\alpha)$$

In general, under analogous conditions, we have

$$\mathcal{F}\left\{f^{(n)}(x)\right\} = (i\alpha)^n F(\alpha)$$

where $n = 0, 1, 2, \ldots$.

It is important to realize that the sine and cosine transforms are not suitable for transforming the first derivatives and in fact any odd-order derivatives:

$$\mathcal{F}_s\left\{f'(x)\right\} = -\alpha\mathcal{F}_c\left\{f(x)\right\} \quad \text{and} \quad \mathcal{F}_c\left\{f'(x)\right\} = \alpha\mathcal{F}_s\left\{f(x)\right\} - f(0)$$

as these are not expressed in terms of the original integral transform.

(ii) Fourier sine transform (optional)

Suppose f and f' are continuous, f is absolutely integrable on $[0, \infty)$ and f'' is piecewise continuous on every finite interval. If $f \to 0$ and $f' \to 0$ as $x \to \infty$, then

$$\begin{aligned}
\mathcal{F}_s\left\{f''(x)\right\} &= \int_0^\infty f''(x)\ \sin\alpha x\,dx = \left[f'(x)\ \sin\alpha x\right]_0^\infty - \alpha\int_0^\infty f'(x)\ \cos\alpha x\,dx \\
&= -\alpha\left[f(x)\ \cos\alpha x\right]_0^\infty - \alpha^2\int_0^\infty f(x)\ \sin\alpha x\,dx = \alpha f(0) - \alpha^2\mathcal{F}_s\{f(x)\} \\
\mathcal{F}_s\left\{f''(x)\right\} &= -\alpha^2 F(\alpha) + \alpha f(0) \qquad (19)
\end{aligned}$$

(iii) Fourier cosine transform (optional)

Under the same assumptions, we find the Fourier the Fourier cosine transform of $f''(x)$ to be

$$\mathcal{F}_c\left\{f''(x)\right\} = -\alpha^2 F(\alpha) - f'(0) \qquad (20)$$

Properties of the Fourier transform

Let us identify time t with the variable x and the angular frequency ω with α. Then the Fourier transform of a function of time $f(t)$, a signal, produces the spectrum of the signal in the representation given by the angular frequency ω.

1. **Linearity**
   The Fourier transform is a linear operator:
   $$\mathcal{F}\{k_1 f_1(t) + k_2 f_2(t)\} = k_1 F_1(\omega) + k_2 F_2(\omega) \tag{21}$$
   where $\mathcal{F}\{f_1(t)\} = F_1(\omega)$ and $\mathcal{F}\{f_2(t)\} = F_2(\omega)$.

2. **Time translation/shifting**
   Time translation or shifting by an amount t_0 leads to a phase shift in the Fourier

transform:

$$\mathcal{F}\left\{f(t-t_0)\right\} = e^{-i\omega t_0}F(\omega) \tag{22}$$

3. **Frequency translation/shifting**

$$\mathcal{F}\left\{e^{i\omega_0 t}f(t)\right\} = F(\omega-\omega_0) \tag{23}$$

The multiplication of $f(t)$ by $e^{i\omega_0 t}$ is called the **complex modulation**. Thus, the complex modulation in the time domain corresponds to a shift in the frequency domain.

4. **Time scaling**

$$\mathcal{F}\left\{f(kt)\right\} = \frac{1}{|k|}\,F\left(\frac{\omega}{k}\right) \tag{24}$$

Therefore if t is directly scaled by a factor k, then the frequency variable is inversely scaled by the factor k. Consequently, for $k > 1$ we have a time-compression resulting in a frequency spectrum expansion. For $k < 1$ there is a time-expansion and a resulting frequency spectrum compression.

5. **Time reversal**

This property follows from the time scaling for $k = -1$

$$\mathcal{F}\left\{f(-t)\right\} = F(-\omega) \tag{25}$$

6. **Symmetry**
   This property is very useful in evaluation of certain Fourier transforms

$$\mathcal{F}\left\{F(t)\right\} = 2\pi f(-\omega) \tag{26}$$

7. **Fourier transform and inverse Fourier transform of a derivative**

$$\mathcal{F}\left\{\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right\} = i\omega F(\omega) \tag{27}$$

$$\tag{28}$$

$$\mathcal{F}^{-1}\left\{\frac{\mathrm{d}F(\omega)}{\mathrm{d}\omega}\right\} = -itf(t) \tag{29}$$

8. **Fourier transform of an integral**

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(u)\, du\right\} = \pi F(0)\delta(\omega) + \frac{1}{i\omega}F(\omega) \tag{30}$$

9. **Fourier transform of a convolution**

$$\mathcal{F}\{f_1(t) * f_2(t)\} = \mathcal{F}\left\{\int_0^t f_1(\tau)\, f_2(t-\tau)\, d\tau\right\} = F_1(\omega)\, F_2(\omega) \tag{31}$$

The counterpart of convolution in the time domain is multiplication in the frequency domain.

10. **Fourier transform of a product**

$$\mathcal{F}\{f_1(t)\, f_2(t)\} = \frac{1}{2\pi} F_1(\omega) * F_2(\omega) \tag{32}$$

Example: Fourier transform of a simple piecewise continuous function

$$f(t) = \begin{cases} -2, & -\pi \leq t < 0 \\ 2, & 0 \leq t < \pi \\ 0, & \text{Otherwise} \end{cases}$$

Solution:

$$\begin{aligned} F(\omega) &= \int_{-\pi}^{0} (-2) e^{-i\omega t}\, dt + \int_{0}^{\pi} (2) e^{-i\omega t}\, dt = \frac{2}{i\omega}\left[e^{-i\omega t}\right]_{-\pi}^{0} - \frac{2}{i\omega}\left[e^{-i\omega t}\right]_{0}^{\pi} \\ &= \frac{2}{i\omega}\left[\left(1 - e^{i\omega\pi}\right) - \left(e^{-i\omega\pi} - 1\right)\right] = \frac{2}{i\omega}\left[2 - 2\cos(\omega\pi)\right] \\ F(\omega) &= \frac{4}{i\omega}\left[1 - \cos(\omega\pi)\right] \end{aligned}$$