Solutions to EE 112 Problem Set 11

P. 1

(a) First we need to find all of the minors of this matrix (9 since it's a 3×3 matrix). Recall that we get M_{ij} by deleting row i and column j, then taking the determinant of the resulting matrix. Thus, for this matrix,

$$M_{11} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 11 & 4 \\ 2 & -1 \end{vmatrix} = (11)(-1) - (4)(2) = -19$$

$$M_{12} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ 0 & -1 \end{vmatrix} = (2)(-1) - (4)(0) = -2$$

$$M_{13} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 11 \\ 0 & 2 \end{vmatrix} = (2)(2) - (11)(0) = 4$$

$$M_{21} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 2 & -1 \end{vmatrix} = (5)(-1) - (2)(2) = -9$$

$$M_{22} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} = (1)(-1) - (2)(0) = -1$$

$$M_{23} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 0 & 2 \end{vmatrix} = (1)(2) - (5)(0) = 2$$

$$M_{31} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 5 & 2 \\ 11 & 4 \end{vmatrix} = (5)(4) - (2)(11) = -2$$

$$M_{32} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = (1)(4) - (2)(2) = 0$$

$$M_{33} = \begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 2 & 11 \end{vmatrix} = (1)(11) - (5)(2) = 1$$

The cofactors C_{ij} are just $(-1)^{i+j} M_{ij}$, so we get

$$C_{11} = (-1)^{1+1} M_{11} = -19, \quad C_{12} = (-1)^{1+2} M_{12} = 2, \quad C_{13} = (-1)^{1+3} M_{13} = 4$$

$$C_{21} = (-1)^{2+1} M_{21} = 9, \quad C_{22} = (-1)^{2+2} M_{22} = -1, \quad C_{23} = (-1)^{2+3} M_{23} = -2$$

$$C_{31} = (-1)^{3+1} M_{31} = -2, \quad C_{32} = (-1)^{3+2} M_{32} = 0, \quad C_{33} = (-1)^{3+3} M_{33} = 1.$$

which gives the cofactor matrix $C = (C_{ij})_{3\times 3}$ to be

$$C = \begin{pmatrix} -19 & 2 & 4 \\ 9 & -1 & -2 \\ -2 & 0 & 1 \end{pmatrix}.$$

The adjugate of the original matrix is the transpose of this:

$$\mathrm{adj}\begin{pmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{pmatrix} = C^T = \begin{pmatrix} -19 & 9 & -2 \\ 2 & -1 & 0 \\ 4 & -2 & 1 \end{pmatrix}$$

All we need now is the determinant of the original matrix; since there's a zero in the 1st column, let's compute the determinant using that:

$$\begin{vmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{vmatrix} = (1)(C_{11}) + (2)(C_{21}) + (0)(C_{31})$$

$$= -19 + 18 = -1$$

So we're done:

$$\begin{pmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{pmatrix}^{-1} = \frac{1}{-1}\,\mathrm{adj}\begin{pmatrix} 1 & 5 & 2 \\ 2 & 11 & 4 \\ 0 & 2 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 19 & -9 & 2 \\ -2 & 1 & 0 \\ -4 & 2 & -1 \end{pmatrix}}$$

(b) We do the same for $\begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{pmatrix}$. The cofactors are

$$C_{11} = (-1)^{1+1}\begin{vmatrix} 0 & 4 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{12} = (-1)^{1+2}\begin{vmatrix} 0 & 4 \\ 2 & 0 \end{vmatrix} = 8, \quad C_{13} = (-1)^{1+3}\begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} = 0$$

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 8 & 0 \\ 0 & 0 \end{vmatrix} = 0, \quad C_{22} = (-1)^{2+2}\begin{vmatrix} 0 & 0 \\ 2 & 0 \end{vmatrix} = 0, \quad C_{23} = (-1)^{2+3}\begin{vmatrix} 0 & 8 \\ 2 & 0 \end{vmatrix} = 16$$

$$C_{31} = (-1)^{3+1}\begin{vmatrix} 8 & 0 \\ 0 & 4 \end{vmatrix} = 32, \quad C_{32} = (-1)^{3+2}\begin{vmatrix} 0 & 0 \\ 0 & 4 \end{vmatrix} = 0, \quad C_{33} = (-1)^{3+3}\begin{vmatrix} 0 & 8 \\ 0 & 0 \end{vmatrix} = 0$$

so

$$\mathrm{adj}\begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \begin{pmatrix} 0 & 0 & 32 \\ 8 & 0 & 0 \\ 0 & 16 & 0 \end{pmatrix}$$

The determinant is

$$\begin{vmatrix} 0 & 8 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{vmatrix} = (0)(C_{11}) + (8)(C_{12}) + (0)(C_{13}) = 64$$

also

$$\begin{pmatrix} 0 & 8 & 0 \\ 0 & 0 & 4 \\ 2 & 0 & 0 \end{pmatrix}^{-1} = \boxed{\frac{1}{64}\begin{pmatrix} 0 & 0 & 32 \\ 8 & 0 & 0 \\ 0 & 16 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1/2 \\ 1/8 & 0 & 0 \\ 0 & 1/4 & 0 \end{pmatrix}.}$$

P.2

(a) How to invert some matrices using GJ reduction: remember the idea is to first create the augmented matrix, which has the original matrix you want to invert in its left $m\times m$ part, and the $m\times m$ identity matrix in the right part. So if $A = \begin{pmatrix} 2 & 3 \\ -2 & 7 \end{pmatrix}$, the augmented matrix is

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -2 & 7 & 0 & 1 \end{array}\right)$$

We want to use row operations to change the 2×2 left part to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; A^{-1} will then be in the right part. So we need to get rid of the -2 in the lower left-hand corner, and we see that adding row 1 to row 2 does just that:

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -2 & 7 & 0 & 1 \end{array}\right) \xrightarrow[\text{(2nd row)}]{\text{(1st row)} +} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -2+(2) & 7+(3) & 0+(1) & 1+(0) \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 10 & 1 & 1 \end{array}\right)$$

If we divide row 2 by 10, we'll get a 1 in the second spot, just as we want:

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 10 & 1 & 1 \end{array}\right) \xrightarrow[\text{by } 10]{\text{divide row } 2} \left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{10} & \frac{1}{10} \end{array}\right)$$

Now, adding (-3) times the 2^{nd} row to the first will get rid of the 3, as we want:

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 0 & 1 & \frac{1}{10} & \frac{1}{10} \end{array}\right) \xrightarrow[+(1^{st}\text{ row})]{(-3)\times(2^{nd}\text{ row})} \left(\begin{array}{cc|cc} 2+(0) & 3+(-3) & 1+(\frac{-3}{10}) & 0+(\frac{-3}{10}) \\ 0 & 1 & \frac{1}{10} & \frac{1}{10} \end{array}\right)$$

$$= \left(\begin{array}{cc|cc} 2 & 0 & \frac{7}{10} & -\frac{3}{10} \\ 0 & 1 & \frac{1}{10} & \frac{1}{10} \end{array}\right)$$

Finally, if we divide row 1 by 2, we'll have $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ in the left 2×2 part, and we'll be done:

$$\left(\begin{array}{cc|cc} 2 & 0 & \frac{7}{10} & -\frac{3}{10} \\ 0 & 1 & \frac{1}{10} & \frac{1}{10} \end{array}\right) \xrightarrow[\text{by } (2)]{\text{divide (row 1)}} \left(\begin{array}{cc|cc} 1 & 0 & \frac{7}{20} & -\frac{3}{20} \\ 0 & 1 & \frac{1}{10} & \frac{1}{10} \end{array}\right)$$

and thus

$$\begin{pmatrix}2&3\\-2&7\end{pmatrix}^{-1} = \boxed{\begin{pmatrix}\frac{7}{20}&-\frac{3}{20}\\\frac{1}{10}&\frac{1}{10}\end{pmatrix} = \frac{1}{20}\begin{pmatrix}7&-3\\2&2\end{pmatrix}}$$

which you can check to see is the correct answer. (Note it agrees with $\begin{pmatrix}a&b\\c&d\end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d&-b\\-c&a\end{pmatrix}$.)

(b) This 3x3 one is a bit longer, so here are the steps:

$$\left(\begin{array}{ccc|ccc}2&0&1&1&0&0\\-3&3&-1&0&1&0\\0&-4&1&0&0&1\end{array}\right) \xrightarrow[\text{by }(2)]{\text{Divide (row 1)}} \left(\begin{array}{ccc|ccc}1&0&\frac12&\frac12&0&0\\-3&3&-1&0&1&0\\0&-4&1&0&0&1\end{array}\right)$$

$$\xrightarrow[\text{to }(2^{nd}\text{ row})]{\text{Add }(3)\times(1^{st}\text{ row})} \left(\begin{array}{ccc|ccc}1&0&\frac12&\frac12&0&0\\0&3&\frac12&\frac32&1&0\\0&-4&1&0&0&1\end{array}\right) \xrightarrow[\text{by }(3)]{\text{Divide (row 2)}} \left(\begin{array}{ccc|ccc}1&0&\frac12&\frac12&0&0\\0&1&\frac16&\frac12&\frac13&0\\0&-4&1&0&0&1\end{array}\right)$$

$$\xrightarrow[\text{to (row 3)}]{\text{add }(4)\times\text{(row 2)}} \left(\begin{array}{ccc|ccc}1&0&\frac12&\frac12&0&0\\0&1&\frac16&\frac12&\frac13&0\\0&0&\frac53&2&\frac43&1\end{array}\right) \xrightarrow[\text{by }(3/5)]{\text{Multiply (3rd row)}} \left(\begin{array}{ccc|ccc}1&0&\frac12&\frac12&0&0\\0&1&\frac16&\frac12&\frac13&0\\0&0&1&\frac65&\frac45&\frac35\end{array}\right)$$

$$\xrightarrow[\text{to (row 2)}]{\text{Add }(-\frac16)\times\text{(row 3)}} \left(\begin{array}{ccc|ccc}1&0&\frac12&\frac12&0&0\\0&1&0&\frac{3}{10}&\frac15&-\frac{1}{10}\\0&0&1&\frac65&\frac45&\frac35\end{array}\right) \xrightarrow[\text{to (row 1)}]{\text{Add }(-\frac12)\times\text{(row 3)}} \left(\begin{array}{ccc|ccc}1&0&0&-\frac{1}{10}&-\frac25&-\frac{3}{10}\\0&1&0&\frac{3}{10}&\frac15&-\frac{1}{10}\\0&0&1&\frac65&\frac45&\frac35\end{array}\right)$$

and so

$$\begin{pmatrix}2&0&1\\-3&3&-1\\0&-4&1\end{pmatrix}^{-1} = \boxed{\begin{pmatrix}-\frac{1}{10}&-\frac25&-\frac{3}{10}\\\frac{3}{10}&\frac15&-\frac{1}{10}\\\frac65&\frac45&\frac35\end{pmatrix} = \frac{1}{10}\begin{pmatrix}-1&-4&-3\\3&2&-1\\12&8&6\end{pmatrix}}.$$

P.3

(a) Here what we need to do is to write m eq'ns in n unknowns of the form $A\underline{X} = B$, where $\underline{X}$ is an $n\times 1$ matrix with the unknowns in its, A is an $m\times n$ matrix consisting of the coefficients of these unknowns, and B is a constant $m\times 1$ matrix giving what's on the right-hand side of the eq'n.

(i) We order our variables x, y, i.e. $\underline{X} = \begin{pmatrix} x \\ y \end{pmatrix}$. The 1^{st} eq'n has 8 as the coefft of x and -2 times y, so the top row of A will be $(8 \;\; 2)$. The second eq'n has $-4x$ and y, so $(-4, 1)$ is the second row. Then,

$$\begin{matrix} 8x-2y \\ -4x+y \end{matrix} \rightarrow \begin{pmatrix} 8 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8x+2y \\ -4x+y \end{pmatrix}$$

But $8x+2y = 1$ and $-4x+y = -10$, so this is $\begin{pmatrix} 1 \\ -10 \end{pmatrix}$. Thus,

$$\boxed{\begin{pmatrix} 8 & -2 \\ -4 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}}$$

is the matrix form we're after.

(ii) Here we have two eq'ns but three unknowns, so if $\underline{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, A will be a 2×3 matrix: Eq'n 1 has $3, 1$ and -1 as the coefficients of x_1, x_2 and x_3, and Eq'n 2 has $1, 0$ and -2. Thus, $A = \begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}$. The 1^{st} eq'n has 20 on the RHS and the second has 16, so $B = \begin{pmatrix} 20 \\ 16 \end{pmatrix}$, giving

$$\boxed{\begin{pmatrix} 3 & 1 & -1 \\ 1 & 0 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 20 \\ 16 \end{pmatrix}}$$

so this is the matrix form.

(iii) This has three eq'ns, three unknowns, and following the same ideas as above, we have

$$\boxed{\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -5 & -4 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ -12 \\ -32 \end{pmatrix}}.$$

(6)

(b) Recall that unique sol'ns generally only exists for systems that have $\underline{\text{exactly}}$ the same number of eq'ns as unknowns; thus, (ii) will not have a unique sol'n (three unknowns but only two eq'ns). (i) and (iii) look good, as they have the same number of eq'ns as unknowns.

But recall that this isn't a guarantee: even if $AX = B$ describes m eq'ns in m unknowns, there's a unique sol'n only if $\det(A) \neq 0$. For (i), we have $\det\begin{pmatrix} 8 & -2 \\ -4 & 1 \end{pmatrix} = 8 - 8 = 0$, so even though there are the same number of eq'ns as unknowns, this will not have a unique sol'n. (In fact, it has $\underline{\text{no}}$ sol'n; no x & y can be found satisfying both $8x - 2y = 1$ and $-4x + y = -10$.)

That leaves (iii), and we need to check the determinant of the coefficient matrix:

$$\begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -5 & -4 & 3 \end{vmatrix} = (2)(-1)^{1+1}\begin{vmatrix} 3 & -1 \\ -4 & 3 \end{vmatrix} + (0)(-1)^{1+2}\begin{vmatrix} 1 & -1 \\ -5 & 3 \end{vmatrix} + (1)(-1)^{1+3}\begin{vmatrix} 1 & 3 \\ -5 & -4 \end{vmatrix}$$

$$= 10 + 0 + 11 = 21$$

which is nonzero, so this is the one that will have a unique sol'n.

$\underline{\text{Inverse Matrix Method:}}$ here we use

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -5 & -4 & 3 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ -12 \\ 32 \end{pmatrix} \Rightarrow \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -5 & -4 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -1 \\ -12 \\ 32 \end{pmatrix}$$

but we need to find the inverse of the coefficient matrix. We computed $\det\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -5 & -4 & 3 \end{pmatrix}$ already, so let's use the adjugate method

$$\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -5 & -4 & 3 \end{pmatrix}^{-1} = \frac{1}{21}\operatorname{adj}\begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & -1 \\ -5 & -4 & 3 \end{pmatrix} = \frac{1}{21}\begin{pmatrix} 5 & 2 & 11 \\ -4 & 11 & 8 \\ -3 & 3 & 6 \end{pmatrix}^T$$

$$= \frac{1}{21}\begin{pmatrix} 5 & -4 & -3 \\ 2 & 11 & 3 \\ 11 & 8 & 6 \end{pmatrix}$$

so

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \frac{1}{21}\begin{pmatrix} 5 & -4 & -3 \\ 2 & 11 & 3 \\ 11 & 8 & 6 \end{pmatrix}\begin{pmatrix} -1 \\ -12 \\ 32 \end{pmatrix} = \frac{1}{21}\begin{pmatrix} -53 \\ -38 \\ 85 \end{pmatrix}$$

so

$$\boxed{u = -\frac{53}{21}, \quad v = -\frac{38}{21}, \quad w = \frac{85}{21}}$$

is the unique sol'n.

Now we use GJ elimination; recall that this starts with the 3×4 matrix where the left-hand 3×3 part is the coefficient matrix A and the right 3×1 part is B, i.e.

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 1 & 3 & -1 & -12 \\ -5 & -4 & 3 & 32 \end{array}\right)$$

We now want to use the same tricks as in P2 to get the LHS into upper-triangular form, and then use this to write down a new set of equivalent eqns that can be easily solved: one will involve only the third variable (w) and can be solved immediately. One will have only v & w, and since we have w, we get v. The last will involve u, v and w, but we follow from v & w. So here we go:

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 1 & 3 & -1 & -12 \\ -5 & -4 & 3 & 32 \end{array}\right) \xrightarrow[+(2^{nd}\text{ row})]{(-\frac{1}{2})\times(1^{st}\text{ row})} \left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 3 & -3/2 & -23/2 \\ -5 & -4 & 3 & 32 \end{array}\right)$$

$$\xrightarrow[+(3^{rd}\text{ row})]{(\frac{5}{2})\times(1^{st}\text{ row})} \left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 3 & -3/2 & -23/2 \\ 0 & -4 & 11/2 & 59/2 \end{array}\right) \xrightarrow[+(3^{rd}\text{ row})]{(\frac{4}{3})\times(2^{nd}\text{ row})} \left(\begin{array}{ccc|c} 2 & 0 & 1 & -1 \\ 0 & 3 & -3/2 & -23/2 \\ 0 & 0 & 7/2 & 85/6 \end{array}\right)$$

and so the new eqns are

$2u + w = -1$, $3v - \frac{3}{2}w = -\frac{23}{2}$, $\frac{7}{2}w = 85/6$.

The last one gives $\boxed{w = 85/21}$. The second has

$$v = \frac{1}{3}\left[\frac{3}{2}w - \frac{23}{2}\right] = \frac{1}{3}\left[\frac{85}{14} - \frac{23}{2}\right] = \boxed{-38/21}$$

and so, finally,

$$u = \frac{1}{2}(-w-1) = \frac{1}{2}\left(-\frac{85}{21} - 1\right) = \boxed{-\frac{53}{21}},$$

exactly as we got before.