Eigenvectors and eigenvalues

Eigenvalue problems

- **From the standpoint of engineering, eigenvalue problems arise very frequently.**

- **Roughly speaking, eigenvalue problems involve finding all non-zero X and all λ**

$$AX = \lambda X$$

- **The set of λ satisfying this equation is called the spectrum of A.**

- **The largest absolute values of the eigenvalues of A is called the spectral radius of A.**

Eigenvalue problems

- **Given any square matrix A, for example:**

$$A = \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix}$$

how do we find the eigenvalues and the eigenvectors of this matrix?

- **Always go back to original problem. We require that**

$$AX = \lambda X \Rightarrow \begin{bmatrix} 4 & 2 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

- **We wish to find values for all the unknowns in the above equation.**

Eigenvalue problems

- **Note that the above matrix equation corresponds to the set of linear equations:**

$$4x_1 + 2x_2 = \lambda x_1$$
$$5x_1 + 1x_2 = \lambda x_2$$

which are easily rewritten as a homogeneous system of linear equations:

$$(4 - \lambda)x_1 + 2x_2 = 0$$
$$5x_1 + (1 - \lambda)x_2 = 0$$

namely:

$$(A - \lambda I)X = 0$$

Eigenvalue problems

- **We seek non-trivial solutions to this set of equations.**
- **Recall that the homogeneous system of equations in n variables and n equations will have a non-trivial solution if and only if the rank of**

$$(A - \lambda I)$$

is less than n. In other words, if and only of this matrix is not-invertible.

- **We know from Cramers rule that this matrix is not invertible if**

$$\det[A - \lambda I] = 0$$

Eigenvalue problems

- **Proceeding along these lines we thus have:**

$$\det[A - \lambda I] = 0$$

$$\Rightarrow \quad \begin{vmatrix} 4-\lambda & 2 \\ 5 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (4-\lambda)(1-\lambda) - (5)(2) = -$$

$$\Rightarrow \quad \lambda^2 - 5\lambda - 6 = 0$$

$$\Rightarrow \quad \lambda = \frac{5 \pm \sqrt{(-5)(-5) - (4)(1)(6)}}{2}$$

$$\lambda = 6, \qquad \lambda = -1$$

- **We have now found the eigenvalues of the matrix. It remains to find the eigenvectors.**

Eigenvalue problems

- **Finding eigenvectors is very easy. Can anybody guess?**

- **To each eigenvalue there corresponds at least one eigenvector (sometime more than this).**

- **How do we do this? Once we have calculated the eigenvalues, we simple insert the corresponding eigenvalue into the equations:**

$$AX = \lambda X$$

- **This gives us a set of equations in n unknowns.**

- **Then we use Gaussian elimination to find the entries of the vector X, and this gives us the eigenvector that we are looking for.**

- **We need to do this for all the eigenvalues.**

Eigenvalue problems

- **For example: in the previous example if we choose**

$$\lambda = 6$$

yields

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

Using Gaussian elimination we find that there is an a nontrivial solution (infinite number of solutions) that satisfy:

$$x_1 = x_2$$

Eigenvalue problems

- **Find the eigenvectors that correspond to the eigenvalue**

$$\lambda = -1$$

- **Show that these correspond to:**

$$-5x_1 = 2x_2$$

Eigenvalue problems

- **Note that it is only the direction (positive or negative) of the eigenvectors (and not the magnitude) that is important. Consequently, as multiplying a vector by any scalar does not change its direction, any vector that is an eigenvector of the matrix A which is multiplied by a positive number, will also be an eigenvector of A.**

$$A(kX) = \lambda(kX)$$

- **This follows from both the definition of an eigenvector and from the fact that we have a set of homogeneous equations when we solve for X.**

Eigenvalue problems

- **Find the eigenvalues and eigenvectors of**

$$(i)\quad \begin{bmatrix} 3 & 4 \\ -1 & 7 \end{bmatrix};$$

$$(ii)\quad \begin{bmatrix} 1 & 2 & 1 \\ 6 & -1 & 0 \\ -1 & -2 & -1 \end{bmatrix}$$

The characteristic equation

- We saw that the eigenvalues of the matrix A are determined by solving the polynomial equation:

$$\det[A - \lambda I] = 0$$

- This equation is very important and is called the characteristic polynomial of the matrix A.

- Since this equation is a polynomial of degree n, it follows that an n by n matrix has at most n eigenvalues.

- Sometimes eigenvalues are real, sometimes complex, and sometimes they repeat. Explain!

The characteristic equation

- **The characteristic polynomial is just a polynomial of degree n.**

$$\det[A - \lambda I] = \lambda^n + a_{n-1}\lambda^{n-1} + \ldots + a_0 = 0$$

- **Every n by n matrix has an associated characteristic polynomial.**
- **Named after Hamilton (the greatest Irish mathematician), is the Cayley-Hamilton theorem.**

[Theorem] If f(λ) is the characteristic polynomial of the matrix A, then f(A)=0.

The characteristic equation

[Example] Show that the following matrix is a root of its characteristic polynomial.

$$A = \begin{bmatrix} 7.3 & 0.2 & -3.7 \\ -11.5 & 1.0 & 5.5 \\ 17.7 & 1.8 & -9.3 \end{bmatrix}$$

Is the characteristic polynomial unique?