Linear equations in matrix form

- A system of linear equations can be represented in matrix form.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_1 &= c_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_2 &= c_2 \\
&\ldots \\
a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nm}x_m &= c_n
\end{aligned}
\quad \Leftrightarrow \quad
\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}
\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}
=
\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}
$$

- Sometimes the system of linear equations is written as an augmented matrix.

$$
\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1m} & c_1 \\ a_{21} & a_{22} & a_{23} & a_{2m} & c_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & a_{nm} & c_n \end{bmatrix}
$$

Example

- Lets solve the following set of equations

$$
\begin{aligned}
x_1 - x_2 + 2x_3 + 3x_4 &= 2 \\
-3x_1 + 6x_2 - 3x_3 - 15x_4 &= -3 \\
5x_1 - 8x_2 - x_3 + 17x_4 &= 9 \\
x_1 + x_2 + 11x_3 + 7x_4 &= -7
\end{aligned}
$$

- Use first equation to remove x_1 from all other equations.

- How do we do this?

Example

- Lets solve the following set of equations

$$
\begin{aligned}
x_1 - x_2 + 2x_3 + 3x_4 &= 2 \\
3x_2 + 9x_3 - 6x_4 &= 3 \\
-3x_2 - 11x_3 + 2x_4 &= -1 \\
2x_2 + 9x_3 + 4x_4 &= 5
\end{aligned}
$$

- Use second equation to remove the second x_2 variable from the latter two equations

- How do we do this?

Example

- Lets solve the following set of equations

$$
\begin{aligned}
x_1 - x_2 + 2x_3 + 3x_4 &= 2 \\
3x_2 + 9x_3 - 6x_4 &= 3 \\
-2x_3 - 4x_4 &= 2 \\
3x_3 + 8x_4 &= 3
\end{aligned}
$$

- Use first equation to remove x_3 from last equations.

- How do we do this?

Example

- Lets solve the following set of equations

$$\begin{aligned} x_1 - x_2 + 2x_3 + 3x_4 &= 2 \\ 3x_2 + 9x_3 - 6x_4 &= 3 \\ -2x_3 - 4x_4 &= 2 \\ x_4 &= 3 \end{aligned}$$

- Now use back-substitution to solve for all variables.

- This process is called **Gaussian elimination**

Gaussian elimination

- By working with matrices things become a lot simpler. We do not need to write the variables each time.
- Instead of manipulating the equations directly we operate on the augmented matrix.
- In the previous example:

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 2 \\ -3 & 6 & 3 & -15 & -3 \\ 5 & -8 & -1 & 17 & 9 \\ 1 & 1 & 11 & 7 & 7 \end{bmatrix}$$

Gaussian elimination

- Step 2: Normalise the second row (to make life easier)

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 2 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & -3 & 11 & 2 & -1 \\ 0 & 2 & 9 & 4 & -5 \end{bmatrix}$$

- Use row 2 to eliminate the second variable from rows 3 and 4.

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 2 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & -2 & -4 & 2 \\ 0 & 0 & 3 & 8 & 3 \end{bmatrix}$$

Gaussian elimination

- Step 3: Normalise the third row (to make life easier)

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 2 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 3 & 8 & 3 \end{bmatrix}$$

- Use row 3 to eliminate the second variable from row 4.

$$\begin{bmatrix} 1 & -1 & 2 & 3 & 2 \\ 0 & 1 & 3 & -2 & 1 \\ 0 & 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 2 & 6 \end{bmatrix}$$

- Use backward substitution to solve system of equation.