8 Inverse Matrix

In this section of we will examine two methods of finding the inverse of a matrix, these are

- The adjoint method.
- Gaussian Elimination.

8.1 Matrix Inverse: The Adjoint Method

We require a couple of definitions before we set out the procedure to find the inverse of a matrix.

8.1.1 Type of Matrix: Cofactor matrix

Definition 8.1 (Cofactor Matrix)**.**
Given a $n \times n$ matrix $\mathbf{A}$. The cofactor matrix $\mathbf{C}$ of $\mathbf{A}$ is the matrix formed by evaluating the cofactors of each entry in $\mathbf{A}$

$$\mathbf{C} = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & & & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}$$

Example 8.1.1 (Cofactor Matrix)**.** Find the cofactor matrix for

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

Solution:
In order to find the cofactor matrix for A we will need the cofactors of each and every entry in A,

$$C_{11} = (-1)^{1+1}M_{11} = (-1)^2 \det \begin{pmatrix} 0 & 0 \\ 1 & -2 \end{pmatrix} = 0$$

$$C_{12} = (-1)^{1+2}M_{12} = (-1)^3 \det \begin{pmatrix} 2 & 0 \\ 1 & -2 \end{pmatrix} = 4$$

$$C_{13} = (-1)^{1+3}M_{13} = (-1)^4 \det \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix} = 2$$

continuing this process (you should check this) we will find

$$C_{21} = 2, \quad C_{22} = 2, \quad C_{23} = 2$$
$$C_{31} = 0, \quad C_{32} = 0, \quad C_{33} = -2$$

and thus the cofactor matrix is

$$\mathbf{C} = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

8.1.2 Adjoint of a matrix

Definition 8.2 (Adjoint of a matrix)**.**
The adjoint of a matrix $\mathbf{A}$ denoted $\mathrm{adj}(\mathbf{A})$ is simply the *transpose of the of the cofactor matrix.* That is, if $\mathbf{C}$ denotes the cofactor matrix of $\mathbf{A}$ then

$$\mathrm{adj}(\mathbf{A}) = \mathbf{C}^\top$$

Example 8.1.2 (The adjoint)**.**
Find the adjoint of the matrix

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

Solution:
We have done all the hard work (finding the cofactor matrix) in the previous example

$$\mathbf{C} = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

thus,

$$\mathrm{adj}(\mathbf{A}) = \mathbf{C}^\top = \begin{pmatrix} 0 & 2 & 0 \\ 4 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}$$

8.1.3 The Inverse: Using the adjoint

We are now ready to state (without proof) a useful theorem which will allow us to compute the inverse of a matrix.

Theorem 8.1.1 (Inverse using the adjoint).
Let A be a $n \times n$ matrix. If $det\boldsymbol{A} \neq 0$, then

$$\boldsymbol{A}^{-1} = \frac{1}{det\boldsymbol{A}} adj\boldsymbol{A}$$

The steps involved in finding an inverse using an the adjoint method for a matrix $\mathbf{A}$

1. Find the determinant of the matrix of interest $\det\mathbf{A}$
   - If $\det\mathbf{A} \neq 0$ then the inverse will exist.
   - If $\det\mathbf{A} = 0$ or matrix isn't square then the inverse will not exist.
2. Find the cofactor matrix $\mathbf{C}$, by finding the cofactor for each element of $\mathbf{A}$.
   - The cofactor of the i^{th}-row j^{th}-column element of $\mathbf{A}$ is

     $$C_{ij} = (-1)^{i+j} M_{ij}$$

     where M_{ij} is the minor.
3. Find the adjoint of $\mathbf{A}$

   $$\text{adj}\mathbf{A} = \mathbf{C}^{\top}$$

4. The inverse is given by

   $$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}} \text{adj}\mathbf{A}$$

Example 8.1.3 (The Inverse).
Find the inverse of

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

using the adjoint method.

Solution:

We have the cofactor matrix and the adjoint of $\mathbf{A}$

$$\mathbf{C} = \begin{pmatrix} 0 & 4 & 2 \\ 2 & 2 & 2 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{and} \quad \text{adj}(\mathbf{A}) = \begin{pmatrix} 0 & 2 & 0 \\ 4 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}$$

We can find the determinant of $\mathbf{A}$ by performing a cofactor expansion about any row or column of $\mathbf{A}$. Picking the third column (as it has two zeros) we have

$$\det\mathbf{A} = a_{13}C_{13} + a_{23}C_{23} + a_{33}C_{33}$$

we have all the cofactors (from the cofactor matrix) thus,

$$\det\mathbf{A} = (0)(2) + (0)(2) + (-2)(-2) = 4.$$

According to our theorem concerning the adjoint and the inverse of a matrix we have

$$\mathbf{A}^{-1} = \frac{1}{\det\mathbf{A}}\,\text{adj}\mathbf{A} \qquad = \frac{1}{4}\begin{pmatrix} 0 & 2 & 0 \\ 4 & 2 & 0 \\ 2 & 2 & -2 \end{pmatrix}$$

and thus,

$$\mathbf{A}^{-1} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 1/2 & 0 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}$$

We can check if this is in fact the inverse

$$\mathbf{A}\mathbf{A}^{-1} = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 1/2 & 0 \\ 1/2 & 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathbf{A}^{-1}\mathbf{A} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1 & 1/2 & 0 \\ 1/2 & 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and thus we have an inverse.

8.2 Matrix Inverse: Gaussian Elimination Method

Another useful method used to find an inverse of matrix involves subjecting our matrix to a series of *elementary row operations*.

8.2.1 Operation: Elementary Row Operations

There are three types of elementary tow operations

1. Add/subtract a multiple of one row to another row.
2. Multiply a row by a constant.
3. Interchange two rows.

Interestingly these elementary row operations have very specific effects on the determinant of a matrix.

Row Operation	**Effect on determinant**
Add a multiple of one row to another row	None
Multiply a row by a constant k	multiplied by k
Interchange two rows	multiplied by -1.

How can this be used to find a determinant for matrix? We can reduce a matrix A to upper triangular form using elementary row operations making it a matrix A'. The determinant of A' is easy to find (as it is triangular the determinant is simply the product of the entries on the diagonal) and relate its determinant to the determinant of A by working back through the row operations that were used in the reduction process.

Example 8.2.1 (The determinant using elementary row operations)**.**

Find the determinant of

$$A = \begin{pmatrix} 2 & 4 & 9 \\ 1 & 2 & 4 \\ 1 & 10 & 7 \end{pmatrix}$$

using elementary row operations.

Solution:

$$\begin{pmatrix} 2 & 4 & 9 \\ 1 & 2 & 4 \\ 1 & 10 & 7 \end{pmatrix} \xrightarrow[\text{(det unchanged)}]{\text{R3 to R3}-\text{R2}} \begin{pmatrix} 2 & 4 & 9 \\ 1 & 2 & 4 \\ 0 & 8 & 3 \end{pmatrix} \xrightarrow[\text{(det } \times -1)]{\text{swap R2 and R3}} \begin{pmatrix} 2 & 4 & 9 \\ 0 & 8 & 3 \\ 1 & 2 & 4 \end{pmatrix} \xrightarrow[\text{(det } \times 2)]{2\times\text{R3}} \begin{pmatrix} 2 & 4 & 9 \\ 0 & 8 & 3 \\ 2 & 4 & 8 \end{pmatrix}$$

$$\xrightarrow[\text{(det unchanged)}]{\text{R3 to R3}-\text{R1}} \begin{pmatrix} 2 & 4 & 9 \\ 0 & 8 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

We now have the matrix A transformed into an upper triangular matrix

$$\mathbf{A}' = \begin{pmatrix} 2 & 4 & 9 \\ 0 & 8 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

the determinant of **A**' is given by the product of the elements on the diagonal

$$\det\mathbf{A}' = (2)(8)(-1) = -16$$

The operations that we conducted on the matrix **A** were

Row Operation	**Effect on determinant**
Add a multiple of one row to another row	det unchanged
Interchange two rows	multiplied det by -1
Multiply a row by 2	multiplied det by 2
Add a multiple of one row to another row	det unchanged

and thus,

$$\det\mathbf{A}' = (-1)(2)\det\mathbf{A}$$

thus,

$$\det\mathbf{A} = 8.$$

8.2.2 Matrix inverse using row operations

We can use these row operations to find the inverse of a matrix, the result that we will use is quoted here without proof.

> If a sequence of elementary row operations on a square matrix **A** can reduce the matrix to the identity matrix **I**, then the same sequence of row operations applied to **I** will result in **I** being transformed to $\mathbf{A}^{-1}$.

Of note is that

- If it's not possible to reduce **A** to **I** using elementary row operations then **A** is not invertible.
- If **A** is invertible then there will be more than one way to reduce it to **I**.

Since we are going to perform the same operations on a given matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We will introduce the following **augmented matrix**, which will allow us to manipulate both matrices at the same time easily

$$\left(\begin{array}{ccc|ccc} a_{11} & a_{12} & a_{13} & 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} & 0 & 0 & 1 \end{array}\right)$$

which is nothing more than both the matrices placed adjacent to one another.

Example 8.2.2 (Inverse using row operations and an augmented matrix)**.**
Find the inverse of

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

using elementary row operations.

Solution:
Step 1: Augment the matrix with the identity matrix

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Step 2: Swap rows (and multiply by a constant if necessary) to ensure that the left side of the augmented matrix will have a "1" in the first row first column entry

$$\left(\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Swap R1 and R2}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

Step 3: Add/subtract multiples of the first row to the second and third row such that the first column of the left sided matrix has zeros beneath the leading "1". In this example there is already a zero beneath the 1 and so we only need to work on the last row

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{\text{Subtract } 2\times\text{R1 from R3}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 & -2 & 1 \end{array}\right)$$

Step 4: Divide/multiply the second row by a constant such that the second row second column element becomes a "1". In this example it is already 1.

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 & -2 & 1 \end{array}\right)$$

Step 5: Add/subtract multiples of the second row to the first and third row such that the only non-zero remaining element in the second column is the "1" on the second row

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 & -2 & 1 \end{array}\right) \xrightarrow{\text{add } 4\times\text{R2 to R3}} \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & -2 & 1 \end{array}\right) \xrightarrow{\text{subtract } 2\times\text{R2 from R1}} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & -2 & 1 \end{array}\right)$$

Step 6: Divide/multiply the third row by a constant such that the third row third column element of becomes a "1". In this example we need to divide the third row by 9

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 9 & 4 & -2 & 1 \end{array}\right) \xrightarrow{\text{R3 } \times 1/9} \left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4/9 & -2/9 & 1/9 \end{array}\right)$$

Step 7: Add/subtract multiples of the third row to the first and second row such that the only non-zero remaining element in the third column is the "1" on the third row

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -4 & -2 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4/9 & -2/9 & 1/9 \end{array}\right) \xrightarrow{\text{add } 4\times\text{ R3 to R1}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/9 & 1/9 & 4/9 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4/9 & -2/9 & 1/9 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/9 & 1/9 & 4/9 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4/9 & -2/9 & 1/9 \end{array}\right) \xrightarrow{\text{subtract } 2\times\text{ R3 from R2}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/9 & 1/9 & 4/9 \\ 0 & 1 & 2 & 1/9 & 4/9 & -2/9 \\ 0 & 0 & 1 & 4/9 & -2/9 & 1/9 \end{array}\right)$$

Step 8: Decompose the augmented matrix. The matrix on the left hand side should be the identity while the matrix on the right is the inverse of the original matrix.

$$\mathbf{A}^{-1} = \begin{pmatrix} -2/9 & 1/9 & 4/9 \\ 1/9 & 4/9 & -2/9 \\ 4/9 & -2/9 & 1/9 \end{pmatrix}$$

Example 8.2.3 (Exercises)**.**
Use elementary row operations to find the inverses of the following matrices

$$\mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} \quad \mathbf{C} = \begin{pmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -2 & -3 \\ 0 & -4 & -3 & 2 \end{pmatrix}$$