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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

EE112

Engineering Mathematics II

Matrices and Matrix Algebra

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Contents

6 Matrices and Matrix Algebra

In this section of the course the aim is to introduce the reader to the concept of a matrix and a number of fundamental operations involving matrices.

6.1 Definition of a matrix and the size of a matrix

Definition 6.1 (Matrix)**.**
A matrix is any rectangular array of numbers, expressions or functions.

Note 6.1.
When referring to more than one matrix use the word "matrices". That is matrices is the plural of matrix. Eg: In this course we will be working with a large variety of matrices; don't panic! we will take it one matrix at a time.

In this course we will work with matrices consisting of numbers and variables exclusively.

Definition 6.2 (Matrix element)**.** The numbers within the matrix are known as the *entries* or *elements* of the matrix.

Example 6.1.1.
The following array of numbers is an example of a matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ -1 & 3 & 0.5 \end{pmatrix}$$

We can say that -1 is an *element of the matrix.* Another property to note is that the matrix is made up of 2 *rows* and 3 *columns*. We can use the number of rows and columns to refer to the *size* of the matrix. The given matrix has a size of 2 rows and 3 columns or in short: the matrix is a 2×3 matrix.

Definition 6.3 (Size of a matrix)**.**
The size of a matrix is referred to by stating the number of rows and the number of columns the matrix consists of. In short if a matrix has m rows and n columns it is said to be a

$$m \times n \qquad \text{matrix}$$

(pronounced " m-by-n matrix ").

In general an $m \times n$ matrix has the form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

The entery/element in the i^{th} row and j^{th} column of the matrix $\mathbf{A}$ is written as a_{ij}. Using this notation the subscript serves as an address, enabling us to refer to a specific element in the matrix. Another useful consequence is that the $m \times n$ matrix $\mathbf{A}$ can be abbreviated to

$$\mathbf{A} = (a_{ij})_{m \times n}\,.$$

6.1.1 Type of matrices: column and row vectors

A matrix being a row vector or column vector is based purely upon the size of the matrix.

Definition 6.4 (column vector)**.**
A matrix with consists of one column and one column only is called a *column vector* or *column matrix.*

Definition 6.5 (row vector)**.**
A matrix with consists of one row and row column only is called a *row vector* or *row matrix.*

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \text{ is a } n \times 1 \textbf{ column} \text{ vector.} \qquad \begin{pmatrix} a_1 & a_2 & \cdots & a_n \end{pmatrix} \text{ is a } 1 \times n \textbf{ row} \text{ vector.}$$

Note 6.2. These matrices take their name from vectors. The same vectors which we met earlier in the course. Row and Column vectors/matrices offer us another way of representing vectors. For example the vector

$$2\,\mathbf{i} - \mathbf{j} + 5\,\mathbf{k}$$

can be equivalently represented as $\langle 2, -1, 5 \rangle$ as we saw earlier or

$$\begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} \text{ is a } 3 \times 1 \textbf{ column} \text{ vector.} \qquad \begin{pmatrix} 2 & -1 & 5 \end{pmatrix} \text{ is a } 1 \times 3 \textbf{ row} \text{ vector.}$$

6.1.2 Type of matrices: square matrices

A matrix being a square matrix is again based solely upon the size of the matrix.

Definition 6.6 (square matrix)**.**
Any $n \times n$ matrix is called a square matrix (or a matrix of order n). That is any matrix that has the *same number of rows as it does columns* is known as a square matrix.

A square matrix of order n has the following general form

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{m2} & \cdots & a_{nn} \end{pmatrix}$$

6.2 Equality between matrices

For two matrices to be considered equal the following must be true

- The two matrices must be the same size.
- The two matrices must have the same elements located at the same place.

We can say this more compactly using our mathematical notation. Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are equal if and only if

$$a_{ij} = b_{ij}, \qquad \text{for all } i \text{ and } j.$$

Example 6.2.1.

$$\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \quad \text{same size, same elements in the same place.}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{matrices aren't the same size.}$$

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \neq \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} \quad \text{elements aren't in the same place.}$$

6.3 The Laws of Matrix Addition

Now that we have a basic understanding of what a matrix is and what it means for two matrices to be equal. We begin to develop the usual operations that we have from regular algebra. Starting with addition and substraction.

6.3.1 Operation: Addition and subtraction

When two matrices are the same size we can add/subtract them. The result of adding/subtracting two matrices is another matrix which has the same size as the initial matrices. The sum/difference of the initial matrices is found by adding/subtracting each of the corresponding elements of the initial matrices.

In short we can write this mathematically as

Definition 6.7 (Addition and Subtraction)**.**
If $\mathbf{A}$ and $\mathbf{B}$ are $m \times n$ matrices, then their sum is

$$\mathbf{A} + \mathbf{B} = (a_{ij} + b_{ij})_{m \times n}$$

and their difference is

$$\mathbf{A} - \mathbf{B} = (a_{ij} - b_{ij})_{m \times n} \qquad \text{or} \qquad \mathbf{B} - \mathbf{A} = (b_{ij} - a_{ij})_{m \times n}$$

Example 6.3.1. Given

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 2 & 4 \\ 1 & 1 & 2 \end{pmatrix}$$

then,

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 1+2 & 2+2 & -1+4 \\ 3+1 & 0+1 & 2+2 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 3 \\ 4 & 1 & 4 \end{pmatrix}$$

and,

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 1-2 & 2-2 & -1-4 \\ 3-1 & 0-1 & 2-2 \end{pmatrix} = \begin{pmatrix} -1 & 0 & -5 \\ 2 & -1 & 0 \end{pmatrix}.$$

Example 6.3.2. If

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 1 & 3 & 0 \\ 2 & 2 & 1 \end{pmatrix}$$

Then $\mathbf{A} + \mathbf{B}$ is not defined, since $\mathbf{A}$ and $\mathbf{B}$ are of different sizes.

6.3.2 Type of matrix: the zero matrix

Definition 6.8 (Zero matrix/Null matrix)**.**
A zero matrix, sometimes referred to as a null matrix, is a matrix whose elements are all zeros. It is typically denoted as

$$\mathbf{0}.$$

A zero matrix can be of any size, when the size of the matrix is required it is typically denoted

$$\mathbf{0}_{m,n}$$

this being a matrix with m-rows and n-columns with all entries being 0.

Example 6.3.3 (Zero matrices)**.**

$$\mathbf{0} = \begin{pmatrix} 0 & 0 \end{pmatrix}, \quad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \mathbf{0}_{2,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbf{0}_{4,4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The zero is the additive identity matrix. That is for any matrix $\mathbf{A}$ we have

$$\mathbf{A} + \mathbf{0} = \mathbf{A} \quad \text{and} \quad \mathbf{0} + \mathbf{A} = \mathbf{A}$$

6.3.3 Operation: Scalar multiplication

If k is a real number, then the *scalar multiple* of a matrix $\mathbf{A}$ is

$$k\mathbf{A} = k \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = (ka_{ij})_{m \times n}$$

Example 6.3.4. Given

$$\mathbf{A} = \begin{pmatrix} 2 & -3 \\ 0 & -1 \end{pmatrix} \quad \text{then} \quad 5\mathbf{A} = \begin{pmatrix} 10 & -15 \\ 0 & -5 \end{pmatrix}$$

6.3.4 Properties of matrix addition

We are now in a position to summarise a number of important properties (which can all be proven from the definitions that we have seen in the previous sections).

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be $m \times n$ matrices and k_1 and k_2 be scalars. Then it can be shown that

(i) The commutative law of addition:

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

(ii) The associative law of addition:

$$\mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}$$

(iii) Factoring scalar multiplication

$$(k_1 k_2)\mathbf{A} = k_1(k_2\mathbf{A})$$

(iv) Scalar multiplication by 1

$$1\mathbf{A} = \mathbf{A}$$

(v) Distributive Law 1:

$$k_1(\mathbf{A} + \mathbf{B}) = k_1\mathbf{A} + k_1\mathbf{B}$$

(vi) Distributive Law 2:

$$(k_1 + k_2)\mathbf{A} = k_1\mathbf{A} + k_2\mathbf{A}$$

6.4 Matrix Multiplication

We have seen the steps involved in multiplying a matrix by a scalar as well as the steps involved in the addition and subtraction of matrices, we now turn our attention to the *multiplication of matrices with one another*.

We begin with a simple example as an illustration.

Example 6.4.1. Given the row vector

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$$

and the column vector

$$\mathbf{B} = \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$$

Their product is defined to be

$$\mathbf{AB} = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} = ((2)(2) + (1)(2) + (1)(3)) = (9).$$

Note 6.3.
This multiplication is effectively the dot product between the two vectors!

6.4.1 Operation: Matrix multiplication

Definition 6.9 (Matric multiplication)**.**
In general, if $\mathbf{A}$ is a $m \times p$ matrix and $\mathbf{B}$ is a $p \times n$ matrix, the the product

$$\mathbf{C} = \mathbf{AB}$$

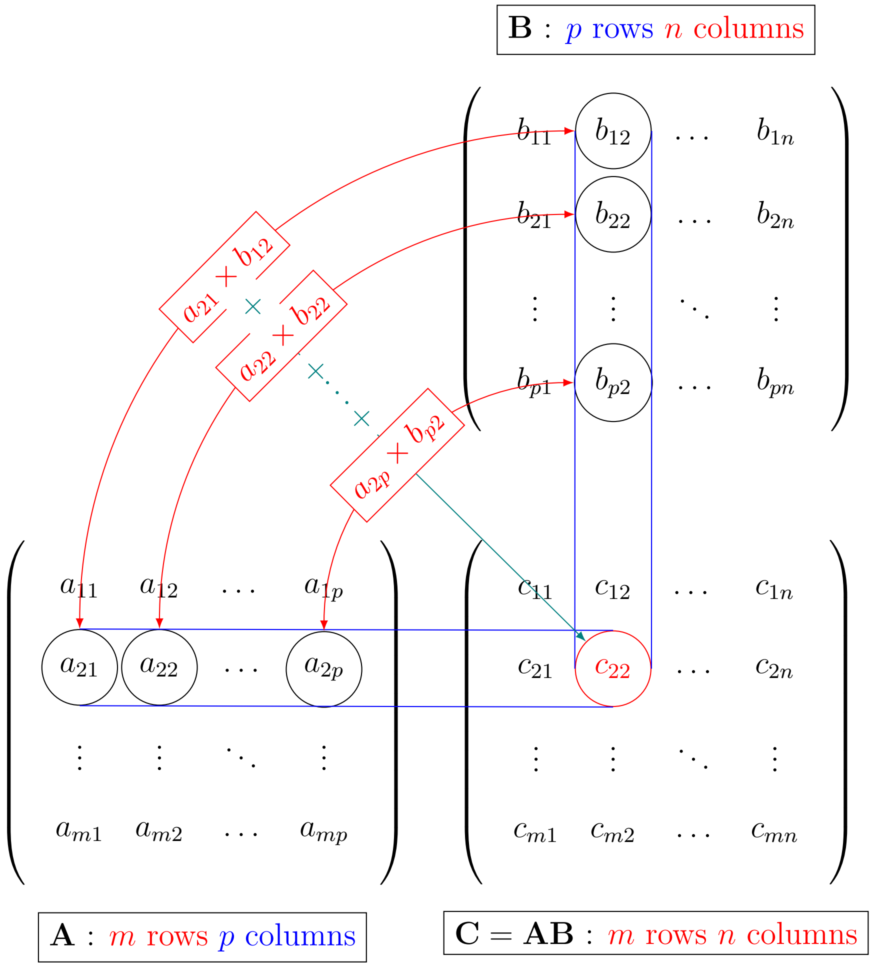
is an $m \times n$ matric whose elements

$$c_{ij}$$

are obtained by "multiplying" the i^{th} row of $\mathbf{A}$ by the j^{th} column of $\mathbf{B}$.

Note 6.4. When looking at the product $\mathbf{AB}$ *the number of columns in matrix* $\boldsymbol{A}$ *must be equal to the number of rows in matrix* $\boldsymbol{B}$. Without this the product $\mathbf{AB}$ does not exist.

Below we have a diagram outlining the procedure for matrix multiplication.



6.4.2 Examples and differences between scalar and matrix multiplication

Example 6.4.2 (Matrix multiplication does not commute.)**.**

If

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}$$

obtain

(i) **AB** (ii)**BA**.

Solution:
(i)

$$\mathbf{AB} = \underset{\color{red}2\color{black}\times\color{blue}2}{\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}} \underset{\color{blue}2\color{black}\times\color{red}2}{\begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix}} = \begin{pmatrix} (1)(2)+(2)(3) & (1)(1)+(2)(-2) \\ (2)(2)+(3)(3) & (2)(1)+(3)(-2) \end{pmatrix} = \underset{\color{red}2\color{black}\times\color{red}2}{\begin{pmatrix} 8 & -3 \\ 13 & -4 \end{pmatrix}}$$

(ii)

$$\mathbf{BA} = \begin{pmatrix} 2 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ -1 & 0 \end{pmatrix}$$

Note 6.5. From this example we can see that

$$\mathbf{AB} \neq \mathbf{BA}$$

That is, matrix multiplication, in general, is not commutative.

Example 6.4.3 (Not all products are defined.)**.**

If

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 4 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

obtain

(i) **AB** (ii)**BA**.

Solution:
(i) First we must check that the product is defined (this is achieved by looking at the size of each matrix):

$$\mathbf{AB} = \underset{\color{red}3\color{black}\times\color{blue}2}{\begin{pmatrix} 3 & 1 \\ 0 & 4 \\ 1 & 2 \end{pmatrix}} \underset{\color{blue}2\color{black}\times\color{red}1}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}$$

match

✓ The matrix multiplication is defined.
The result will be a 3×1 matrix.

Working out the product

$$\mathbf{AB} = \begin{pmatrix} (3)(2) + (1)(1) \\ (0)(2) + (4)(1) \\ (1)(2) + (2)(1) \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 4 \end{pmatrix}.$$

(ii) Check if the multiplication is defined for the given matrices:

$$\mathbf{BA} = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{2 \times 1} \underbrace{\begin{pmatrix} 3 & 1 \\ 0 & 4 \\ 1 & 2 \end{pmatrix}}_{3 \times 2}$$

don't match

$\times$ The matrix multiplication is not defined. There is no product.

$\diamondsuit$

There are a number of peculiarities when working with matrix multiplication. We have already seen that the matrix multiplication is not commutative that is, in general

$$\mathbf{AB} \neq \mathbf{BA}$$

In the next two examples we will see two more interesting differences between matrix multiplication and scalar multiplication.

Example 6.4.4 (Product resulting in zeros)**.**

Compute the product $\mathbf{YZ}$ given

$$\mathbf{Y} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{Z} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Solution:
Since the number of rows of matrix $\mathbf{Y}$ is equal to the number of columns of matrix $\mathbf{Z}$ we know that the matrix multiplication is defined.

$$\mathbf{YZ} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} (1)(-1) + (1)(1) & (1)(1) + (1)(-1) \\ (2)(-1) + (2)(1) & (2)(1) + (2)(-1) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

Thus we have

$$\mathbf{YZ} = \mathbf{0}.$$

Notice however that *neither* $\mathbf{Y}$ nor $\mathbf{Z}$ are themselves a zero matrix.

This is in contrast to scalar multiplication where given two scalars k_1 and k_2, such that

$$k_1 k_2 = 0$$

then either $k_1 = 0$ or $k_2 = 0$. This is *not* the case for matrix multiplication! Thus, given two matrices $\mathbf{A}$ and $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{0}$$

in general **does not imply** that $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.

◇

Example 6.4.5 (Different matrices same product)**.**

Provided with the following three matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{C} = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$$

Show that $\mathbf{AB} = \mathbf{AC}$.

Solution:

$$\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} (1)(2) + (1)(2) & (1)(1) + (1)(2) \\ (2)(2) + (2)(2) & (2)(1) + (2)(2) \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix}$$

and

$$\mathbf{AC} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} (1)(3) + (1)(1) & (1)(0) + (1)(3) \\ (2)(3) + (2)(1) & (2)(0) + (2)(3) \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 8 & 6 \end{pmatrix}$$

Notice however that even though we have

$$\mathbf{AB} = \mathbf{AC}$$

this does not mean that $\mathbf{B} = \mathbf{C}$. This is in complete contrast with scalar multiplication; given scalars k_1, k_2 and k_3 such that $k_1 k_2 = k_1 k_3$ we can conclude that $k_2 = k_3$. However this does not hold with matrix multiplication! We have not met the idea of an *invertible* matrix yet however for later reference, the only situation where $\mathbf{B} \neq \mathbf{C}$ given that $\mathbf{AB} = \mathbf{AC}$ occurs when the matrix $\mathbf{A}$ is *non-invertible.*

$\diamondsuit$

6.4.3 Type of matrix: the identity matrix

Definition 6.10 (Identity matrix)**.**
An identity matrix is a matrix which satisfies all of the following properties:

- The matrix is square.
- The elements on the *main diagonal* are all 1s.
- The elements not on the main diagonal are all 0s.

An identity matrix is typically denoted as

$$\mathbf{I} \quad \text{or sometimes} \quad \mathbb{1}$$

When the size of the identity matrix is required a subscript is typically used such as

$$\mathbf{I}_n$$

this a matrix with n-rows and n-columns, the elements on the main diagonal ($\searrow$) are all ones while every other element is zero.

Example 6.4.6 (Identity matrices)**.**

$$\mathbf{I} = \begin{pmatrix} 1 \end{pmatrix}, \quad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The identity matrix is actually the multiplicative identity matrix. That is for any matrix $\mathbf{A}$ we have

$$\mathbf{IA} = \mathbf{AI} = \mathbf{A}$$

It is important to note however, if $\mathbf{A}$ is an $m \times n$ matrix then we have

$$\mathbf{I}_m\mathbf{A} = \mathbf{A}\mathbf{I}_n = \mathbf{A}$$

The matrices $\mathbf{I}_m$ and $\mathbf{I}_n$ are indeed both identity matrices however they are *different* identity matrices. In this regard one needs to be aware that for non-square matrices $\mathbf{A}$ the 'multiplicative identity' is not just a single matrix. This is unlike the scalar multiplication where the multiplicative identity is simply the number 1. The multiplicative identity for matrices depends on the matrix multiplication at hand.

6.4.4 Properties of matrix multiplication

Given arbitrary matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and a scalar k. Whenever the relevant addition and multiplication is defined we have

(i) The associative law

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

(ii) The left distribution law

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

(iii) The right distribution law

$$(\mathbf{B} + \mathbf{C})\mathbf{A} = \mathbf{BA} + \mathbf{CA}$$

(iv) Scalar multiplication,

$$k(\mathbf{AB}) = (k\mathbf{A})\mathbf{B} = \mathbf{A}(k\mathbf{B}).$$

Here are some important differences between matrix multiplication and scalar multiplication

Given arbitrary matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$

- $\mathbf{AB} \neq \mathbf{BA}$ The order matters.
- If $\mathbf{AB} = \mathbf{0}$ this does not mean that either $\mathbf{A} = \mathbf{0}$ or $\mathbf{B} = \mathbf{0}$.
- If we have matrices such that

$$\mathbf{AB} = \mathbf{AC}$$

  the does not mean that $\mathbf{B} = \mathbf{C}$. We have to first find out if the matrix $\mathbf{A}$ is *invertible* (we will see this in a later section).

6.5 Powers of a matrix and polynomials in matrices

With a clear definition of matrix multiplication we can consider raising matrices to scalar powers. At this point we will simply introduce the basics however we will revisit this topic when we have met *diagonalisation* and again when we have met the *Cayley-Hamilton theorem* in later sections of this course.

6.5.1 Operation: Matrix raised to a power

Given a *square* matrix $\mathbf{A}$ and a natural number k, i.e. $k \in \mathbb{N}$ then

$$\mathbf{A}^k = \underbrace{\mathbf{A} \cdot \mathbf{A} \cdot \ldots \cdot \mathbf{A}}_{k \text{ times.}}$$

Example 6.5.1 (Raising a matrix to a power)**.**

Given the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix}$$

Find (i) $\mathbf{A}^2$, (ii) $\mathbf{A}^3$, (iii) $\mathbf{A}^5$ and $\mathbf{A}^{10}$.

Solution:

(i) To find $\mathbf{A}^2$ we simply multiply $\mathbf{A}$ by itself

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 6 & 4 \end{pmatrix}$$

(ii) To find $\mathbf{A}^3$ we could multiply $\mathbf{A}$ by itself three times or use the expression we found for $\mathbf{A}^2$ in the previous part

$$\mathbf{A}^3 = \mathbf{A}^2\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 14 & 8 \end{pmatrix}$$

(iii) now $\mathbf{A}^5 = \mathbf{A}^3\mathbf{A}^2$

$$\mathbf{A}^5 = \mathbf{A}^3\mathbf{A}^2 = \begin{pmatrix} 1 & 0 \\ 14 & 8 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 6 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 62 & 32 \end{pmatrix}$$

(iv) finally we have

$$\mathbf{A}^{10} = \mathbf{A}^5\mathbf{A}^5 = \begin{pmatrix} 1 & 0 \\ 62 & 32 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 62 & 32 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2046 & 1024 \end{pmatrix}$$

$\diamondsuit$

Here it is worthwhile to make a number a couple of important notes. The powers that we can examine in this manner are *positive integer powers*. We cannot find the result of a matrix raised to a negative power and even through it is possible to define what is meant by raising a matrix to a fractional power we will not be examining this here.

We can however state that for any given square matrix $\mathbf{A}$

$$\mathbf{A}^0 = \mathbf{I}.$$

That is, raising a matrix to the power of zero results in the identity matrix.

It is again worthwhile reminding the reader that matrix algebra can be a little different to what we are used to.

Example 6.5.2.

Given two arbitrary square matrices of order n $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A}^2 - \mathbf{B}^2 = (\mathbf{A} - \mathbf{B})(\mathbf{A} + \mathbf{B}) \quad ?$$

Give a reason for your answer.

Solution:
The answer is no. To see this we will multiply the expression on the righthand side

$$
\begin{aligned}
(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B}) &= \mathbf{A}(\mathbf{A}+\mathbf{B})-\mathbf{B}(\mathbf{A}+\mathbf{B}) \\
&= \mathbf{A}^2+\underbrace{\mathbf{AB}-\mathbf{BA}}_{\neq \mathbf{0}}+\mathbf{B}^2
\end{aligned}
$$

we know however that for matrices in general $\mathbf{AB} \neq \mathbf{BA}$ and thus in general

$$(\mathbf{A}-\mathbf{B})(\mathbf{A}+\mathbf{B}) \neq \mathbf{A}^2-\mathbf{B}^2$$

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Example 6.5.3.

Given two arbitrary square matrices of order n $\mathbf{A}$ and $\mathbf{B}$ is

$$\mathbf{A}^2\mathbf{B}^2 = (\mathbf{AB})^2 \quad ?$$

Give a reason for your answer.

Solution:
In general this above identity does not hold. To see why we examine

$$(\mathbf{AB})^2 = \mathbf{ABAB}.$$

We also have

$$\mathbf{A}^2\mathbf{B}^2 = \mathbf{AABB}.$$

The problem asks if

$$\mathbf{A}^2\mathbf{B}^2 = (\mathbf{AB})^2 \quad ?$$

we can say in general this is not true as

$$\mathbf{ABAB} \neq \mathbf{AABB}$$

by virtue of $\mathbf{AB} \neq \mathbf{BA}$ with matrix multiplication.

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6.5.2 Polynomials in matrices

Though not used extensively in this course, polynomials involving matrices can be found in many aspects of Engineering. Polynomials in matrices are defined as

$$P(\mathbf{X}) = a_0\mathbf{I} + a_1\mathbf{X} + a_2\mathbf{X}^2 + \cdots + a_n\mathbf{X}^n$$

If we can find a matrix $\mathbf{R}$ such that $P(\mathbf{R}) = \mathbf{0}$ then we say that $\mathbf{R}$ is a **root** of the polynomial $P(\mathbf{X})$.

Example 6.5.4 (Matrix polynomial)**.**

Suppose

$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$$

and the polynomials f and g are given by

(i) $f(\mathbf{X}) = 5 + 2\mathbf{X} + 3\mathbf{X}^2$

(ii) $g(\mathbf{X}) = \mathbf{X} + \mathbf{X}^2$

Evaluate $f(\mathbf{A})$ and $g(\mathbf{A})$.

Solution:
(i)

$$\begin{aligned}
f(\mathbf{A}) &= 5\mathbf{I} + 2\mathbf{A} + 3\mathbf{A}^2 \\
&= 5\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} + 3\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} + \begin{pmatrix} 2 & -4 \\ -4 & 6 \end{pmatrix} + 3\begin{pmatrix} 5 & -8 \\ -8 & 14 \end{pmatrix} \\
&= \begin{pmatrix} 22 & -28 \\ -28 & 53 \end{pmatrix}.
\end{aligned}$$

(ii)

$$g(\mathbf{A}) = \mathbf{A} - \mathbf{A}^2 = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix} - \begin{pmatrix} 5 & -8 \\ -8 & 14 \end{pmatrix} = \begin{pmatrix} -4 & 10 \\ 6 & -11 \end{pmatrix}$$

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