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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

EE112

Engineering Mathematics II

Curves and Curvature

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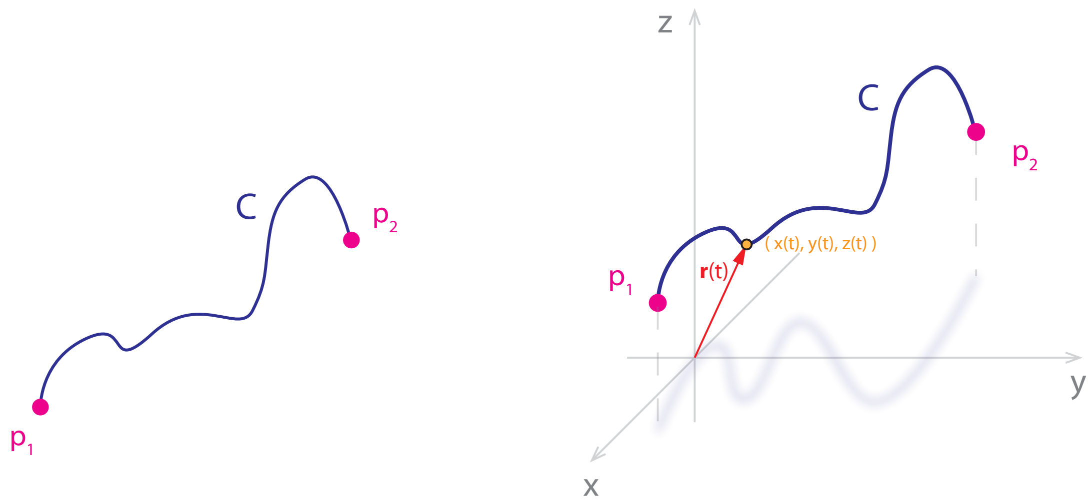
5 Curves and Curvature

Here the aim is to examine curves that are written in parametric form. This section will culminate to an idea of *curvature* of a curve at every point along its length.

5.1 Parametric Representation of a Curve

Consider a curve C



(a) The curve C

(b) At each point on the curve there is an associated position vector

Figure 1: A curve and a position vector associated with each point on the curve.

at a particular point on this curve (x, y, z) one can associate a position vector

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

The collection of position vectors which sweep out the curve C can be written in parametric form as

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

From this parametrised position vector we have the curves *parametric representation*

$$x = x(t), \quad y = y(t) \quad \text{and} \quad z = z(t)$$

5.2 Tangent to a Curve

The usefulness of having the parametric representation can be immediately appreciated when looking for a tangent to a curve; this is given in form of the *tangent vector*

Definition 5.1.

Given a curve C with an associated parametrised position vector

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

The **Tangent vector** is given by

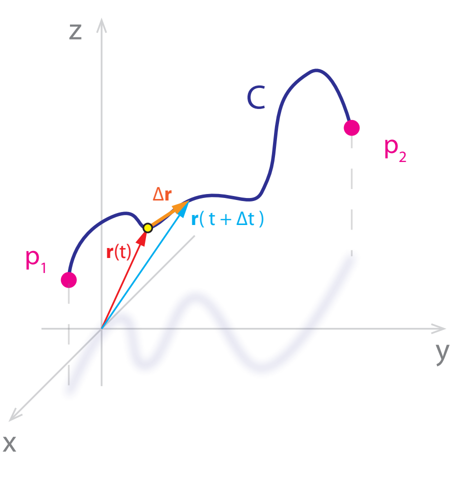
$$\mathbf{r}' \equiv \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$$

where

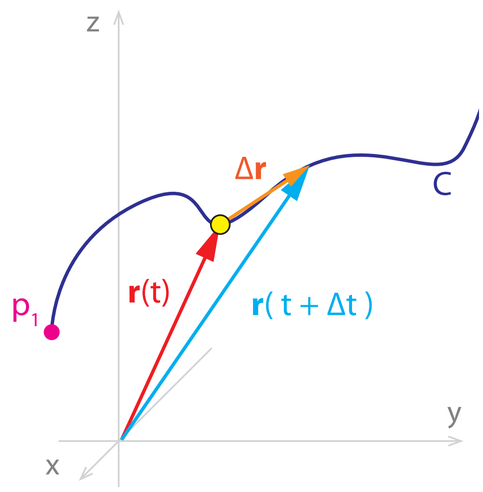
$$x = x(t), \quad y = y(t) \quad \text{and} \quad z = z(t)$$

is the parametric form of the curve.

One can see the result with the aid of a diagram and a little calculus



(a) Looking at position vector to a point on the curve at t and $t + \Delta t$



(b) The vector $\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$

Figure 2: Finding the tangent to the curve at the point with position vector $\mathbf{r}(t)$.

We have

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

the vector $\Delta\mathbf{r}$ is our approximation (made sufficiently small) to the curve at the point with position vector $\mathbf{r}(t)$. We can say that

$$\frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t}$$

to find the tangent to the curve all one requires is to take the limit as $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t+\Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

5.2.1 Unit tangent vector

We can find a unit tangent vector to the curve C by dividing the tangent vector by its length

$$\hat{\mathbf{u}} \equiv \frac{d\mathbf{r}}{dt} \bigg/ \left|\frac{d\mathbf{r}}{dt}\right|$$

5.2.2 Worked example

Example 5.2.1. Consider the vector function

$$\mathbf{r}(t) = a\cos(t)\,\mathbf{i} + b\sin(t)\,\mathbf{j}$$

Identify the curve that the position vector sweeps out. Assume that $a = b$ and find a unit tangent vector to the curve.

Solution:
We have

$$\mathbf{r}(t) = a\cos(t)\,\mathbf{i} + b\sin(t)\,\mathbf{j}$$

but we know that the position vector is given by

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$$

giving us the parameteric equation to the curve

$$x = a\cos(t), \quad \text{and} \quad y = b\sin(t)$$

example continued ...

notice that we can write the parameteric equations as

$$\frac{x}{a} = \cos(t), \quad \text{and} \quad \frac{y}{b} = \sin(t)$$

and hence,

$$\begin{aligned}\frac{x^2}{a^2} + \frac{y^2}{b^2} &= \cos^2(t) + \sin^2(t) \\ &= 1\end{aligned}$$

Thus the curve can be written in *implicit form* as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is the equation of an **ellipse** on the xy plane (z = 0).

We now set about finding a unit tangent vector. The question tells us to assume that $a = b$ which corresponds to the equation of a circle. We know that a tangent to the curve is given by

$$\begin{aligned}\mathbf{u} = \frac{d\mathbf{r}}{dt} &= \frac{d}{dt}[a\cos(t)]\,\mathbf{i} + \frac{d}{dt}[b\sin(t)]\,\mathbf{j} \\ &= -a\sin(t)\,\mathbf{i} + b\cos(t)\,\mathbf{j}.\end{aligned}$$

Letting $b = a$ we have

$$\mathbf{u} = -a\sin(t)\,\mathbf{i} + a\cos(t)\,\mathbf{j}$$

To make the vector of unit length we need to divide by its length

$$|\mathbf{u}| = \sqrt{a^2\sin^2(t) + a^2\cos^2(t)} = \sqrt{a^2(\sin^2(t) + \cos^2(t))} = \sqrt{a^2} = a.$$

The unit tangent vector is given by

$$\hat{\mathbf{u}} = \frac{1}{a}\left(-a\sin(t)\,\mathbf{i} + a\cos(t)\,\mathbf{j}\right) = -\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j}$$

5.3 Arc Length of a Curve

If S is the arc length of the curve C between the points a and b as shown in the diagram

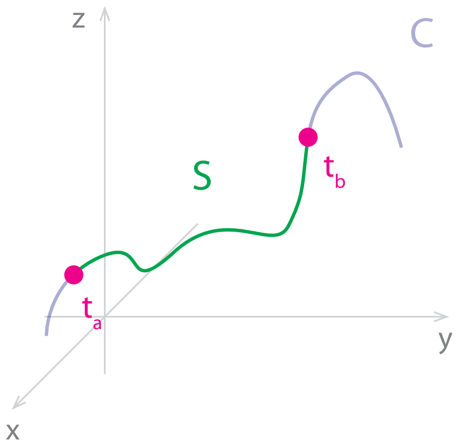


Figure 3: The length of the curve S between two points on the curve C.

Using the notation

$$\mathbf{r}' = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$$

we know that

$$|\mathbf{r}'|^2 = \mathbf{r}' \cdot \mathbf{r}' = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \left(\frac{dS}{dt}\right)^2$$

where $dS^2 = dx^2 + dy^2 + dz^2$. Hence we have an expression for an infinitesimal length along the curve C

$$dS = \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, dt$$

In order to find the length of the section S between the points a and b we simply take the sum of the infinitesimal lengths between the points along the curve,

Important Formula 5.1 (Arc Length of a Curve)**.**

$$S = \int_{t_a}^{t_b} \sqrt{\mathbf{r}' \cdot \mathbf{r}'}\, dt = \int_{t_a}^{t_b} |\mathbf{r}'|\, dt$$

5.4 Curvature and the Principal Unit Normal Vector

We begin with

$$\hat{\mathbf{u}} \cdot \hat{\mathbf{u}} = 1$$

Using the product rule and differentiating with respect to S

$$\hat{\mathbf{u}} \cdot \frac{d\hat{\mathbf{u}}}{dS} + \frac{d\hat{\mathbf{u}}}{dS} \cdot \hat{\mathbf{u}} = 0$$

and now,

$$2\hat{\mathbf{u}} \cdot \frac{d\hat{\mathbf{u}}}{dS} = 0$$

$$\Rightarrow \hat{\mathbf{u}} \cdot \frac{d\hat{\mathbf{u}}}{dS} = 0$$

Hence we have $\frac{d\hat{\mathbf{u}}}{dS}$ is perpendicular to $\hat{\mathbf{u}}$.

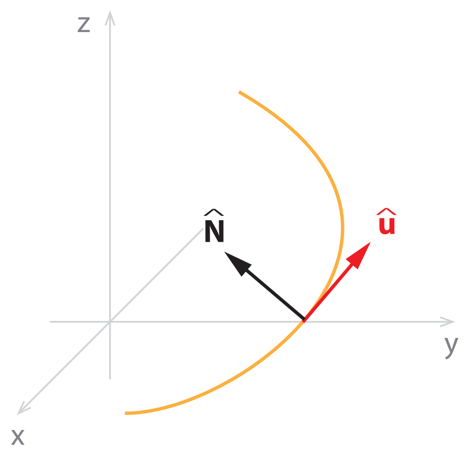


Figure 4: The vector $\hat{\mathbf{N}}$ is perpendicular to $\hat{\mathbf{u}}$.

We can write

$$\frac{d\hat{\mathbf{u}}}{dS} = \kappa\hat{\mathbf{N}}$$

where κ is the curvature.

5.4.1 Direct Calculation of Curvature and unit Normal in terms of ‘t’

We can calculate κ in terms of t without having to calculate $\hat{\mathbf{u}}$ in terms of S first.

$$\kappa \equiv \left|\frac{d\hat{\mathbf{u}}}{dS}\right|$$

now,

$$\frac{d\hat{\mathbf{u}}}{dt} = \frac{d\hat{\mathbf{u}}}{dS}\frac{dS}{dt}$$

hence,

$$\frac{d\hat{\mathbf{u}}}{dS} = \frac{d\hat{\mathbf{u}}}{dt}\Bigg/\frac{dS}{dt}\ .$$

Let,

$$S = \int |\mathbf{r}'|\, dt \Rightarrow \frac{dS}{dt} = |\mathbf{r}'|$$

$$\boxed{\kappa = \frac{\left|\frac{d\hat{\mathbf{u}}}{dt}\right|}{|\mathbf{r}'|} = \left|\frac{d\hat{\mathbf{u}}}{dt}\right|\Bigg/|\mathbf{r}'|\ .}$$

Similarly we can obtain $\hat{\mathbf{N}}$ in terms of t.

$$\hat{\mathbf{N}} = \frac{1}{\kappa}\frac{d\hat{\mathbf{u}}}{dS}$$

i.e.

$$\hat{\mathbf{N}} = \frac{1}{\left|\frac{d\hat{\mathbf{u}}}{dt}\right|\Big/|\mathbf{r}'|}\cdot\frac{\frac{d\hat{\mathbf{u}}}{dt}}{\frac{dS}{dt}} = \frac{\cancel{|\mathbf{r}'|}}{\left|\frac{d\hat{\mathbf{u}}}{dt}\right|}\cdot\frac{\frac{d\hat{\mathbf{u}}}{dt}}{\cancel{|\mathbf{r}'|}}$$

$$\boxed{\hat{\mathbf{N}} = \frac{d\hat{\mathbf{u}}}{dt}\Bigg/\left|\frac{d\hat{\mathbf{u}}}{dt}\right|}$$

Hence we can calculate $\hat{\mathbf{N}}$ without having to calculate S first.

5.4.2 Worked Examples

Example 5.4.1. Calculate

1. the principal unit normal vector, $\hat{\mathbf{N}}$
2. the curvature, κ
3. the radius of curvature, R, for the helix

$$\mathbf{r} = a\cos(t)\,\mathbf{i} + a\sin(t)\,\mathbf{j} + ct\,\mathbf{k}$$

where a and c are positive constants

Solution:

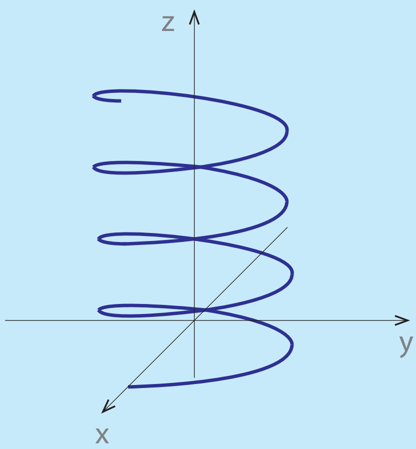


Figure 5: A helix

To find the unit normal we begin by finding the unit tangent vector, $\hat{\mathbf{u}}$

$$\begin{aligned}
\mathbf{r} &= a\cos(t)\,\mathbf{i} + a\sin(t)\,\mathbf{j} + ct\,\mathbf{k} \\
\mathbf{r}' &= -a\sin(t)\,\mathbf{i} + a\cos(t)\,\mathbf{j} + c\,\mathbf{k} \\
\Rightarrow |\mathbf{r}'| &= \sqrt{a^2\cos^2(t) + a^2\sin^2(t) + c^2} = \sqrt{a^2 + c^2}
\end{aligned}$$

hence,

$$\hat{\mathbf{u}} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{1}{\sqrt{a^2 + c^2}}\left\{-a\sin(t)\,\mathbf{i} + a\cos(t)\,\mathbf{j} + c\,\mathbf{k}\right\}$$

example continued...

The normal vector is the derivative of the tangent vector with respect to t

$$\frac{d\hat{\mathbf{u}}}{dt} = \frac{1}{\sqrt{a^2+c^2}}\left\{-a\cos(t)\,\mathbf{i} - a\sin(t)\,\mathbf{j}\right\}$$

which has a length

$$\left|\frac{d\hat{\mathbf{u}}}{dt}\right| = \frac{1}{\sqrt{a^2+c^2}}\left\{\sqrt{a^2\cos^2(t) + a^2\sin^2(t)}\right\} = \frac{a}{\sqrt{a^2+c^2}}.$$

Finally the unit normal vector, $\hat{\mathbf{N}}$, is given by

$$\hat{\mathbf{N}} = \frac{d\hat{\mathbf{u}}}{dt}\Bigg/\left|\frac{d\hat{\mathbf{u}}}{dt}\right| = \frac{1}{\sqrt{a^2+c^2}}\left\{-a\cos(t)\,\mathbf{i} - a\sin(t)\,\mathbf{j}\right\}\frac{\sqrt{a^2+c^2}}{a}$$

$$\hat{\mathbf{N}} = -\cos(t)\,\mathbf{i} - \sin(t)\,\mathbf{j}$$

We now find the curvature κ of the helix, recall that

$$\kappa = \left|\frac{d\hat{\mathbf{u}}}{dt}\right|\Bigg/|\mathbf{r}'| = \frac{a}{\sqrt{a^2+c^2}}\frac{1}{\sqrt{a^2+c^2}}$$

$$\kappa = \frac{a}{a^2+c^2}$$

The radius of curvature $R = \dfrac{1}{\kappa}$

$$R = \frac{1}{\kappa} = \frac{a^2+c^2}{a}$$

Note that when $c = 0$ the helix collapses to a circle in the x-y plane and that the radius of curvature R becomes $R = a$ as would be expected.

Example 5.4.2. Obtain

1. the unit tangent vector, $\hat{\mathbf{u}}$
2. the unit normal vector, $\hat{\mathbf{N}}$
3. the curvature, κ

for the curve,

$$\mathbf{r} = (\cos(t) + t\sin(t))\,\mathbf{i} + (\sin(t) - t\cos(t))\,\mathbf{j}, \qquad t > 0$$

Solution:

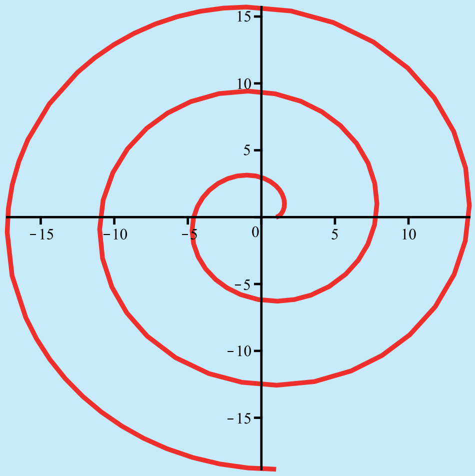


Figure 6: A plot of the curve for $0 \leq t \leq 6\pi$

Taking the first derivative we have the tangent vector

$$\begin{aligned}\mathbf{r}' &= \Big(-\sin(t) + t\cos(t) + \sin(t)\Big)\,\mathbf{i} + \Big(\cos(t) + t\sin(t) - \cos(t)\Big)\,\mathbf{j} \\ &= t\cos(t)\,\mathbf{i} + t\sin(t)\,\mathbf{j}\end{aligned}$$

with length

$$|\mathbf{r}'| = \sqrt{t^2\cos^2(t) + t^2\sin^2(t)} = t$$

hence

$$\hat{\mathbf{u}} = \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j}$$

example continued . . .

The unit normal vector is given by

$$\hat{\mathbf{N}} = \frac{d\hat{\mathbf{u}}}{dt} \Bigg/ \left|\frac{d\hat{\mathbf{u}}}{dt}\right|$$

and we have

$$\frac{d\hat{\mathbf{u}}}{dt} = -\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j}$$

which has a length

$$\left|\frac{d\hat{\mathbf{u}}}{dt}\right| = \sin^2(t) + \cos^2(t) = 1$$

hence,

$$\hat{\mathbf{N}} = -\sin(t)\,\mathbf{i} + \cos(t)\,\mathbf{j}.$$

Finally we find the curvature κ

$$\kappa = \left|\frac{d\hat{\mathbf{u}}}{dt}\right| \Bigg/ |\mathbf{r}'| = \frac{1}{t}$$

Example 5.4.3. Obtain

1. the unit tangent vector, $\hat{\mathbf{u}}$
2. the unit normal vector, $\hat{\mathbf{N}}$
3. the curvature, κ

for the curve,

$$\mathbf{r} = t\,\mathbf{i} + \frac{t^2}{2}\,\mathbf{j}$$

Solution:
For a sketch of the curve recall that

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

which for this curve implies that

$$x = t, \quad y = \frac{t^2}{2}, \quad \text{and} \quad z = 0$$

thus the curve describes the parabola

$$y = \frac{x^2}{2}$$

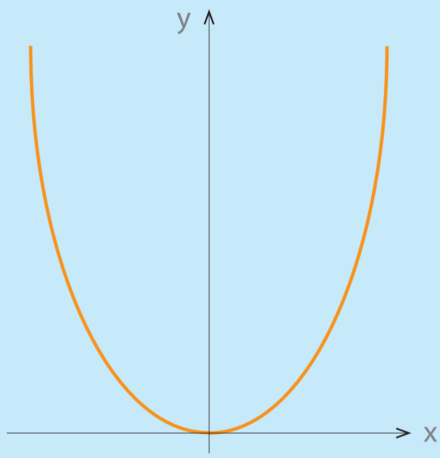


Figure 7: A parabola in the xy-plane.

example continued ...

Now,

$$\mathbf{r} = t\,\mathbf{i} + \frac{t^2}{2}\,\mathbf{j} \Rightarrow \mathbf{r}' = 1\,\mathbf{i} + t\,\mathbf{j}$$

and,

$$|\mathbf{r}'| = \sqrt{1+t^2}$$

Hence,

$$\hat{\mathbf{u}} = \frac{1}{\sqrt{1+t^2}}\left\{1\,\mathbf{i} + t\,\mathbf{j}\right\}$$

To find the normal vector we require the derivative of the unit tangent vector

$$\begin{aligned}
\frac{d\hat{\mathbf{u}}}{dt} &= \frac{1}{\sqrt{1+t^2}}\left\{0\,\mathbf{i} + 1\,\mathbf{j}\right\} + \left\{1\,\mathbf{i} + t\,\mathbf{j}\right\}\left(-\frac{1}{2}(1+t^2)^{-\frac{3}{2}}2t\right) \\
&= \frac{\mathbf{j}}{\sqrt{1+t^2}} - \frac{t(1\,\mathbf{i} + t\,\mathbf{j})}{(1+t^2)^{\frac{3}{2}}} \\
&= \frac{(1+t^2)\,\mathbf{j} - t\,\mathbf{i} - t^2\,\mathbf{j}}{(1+t^2)^{\frac{3}{2}}} \\
&= \frac{1}{(1+t^2)^{\frac{3}{2}}}\left\{-t\,\mathbf{i} + \mathbf{j}\right\}
\end{aligned}$$

the length of this vector is

$$\left|\frac{d\hat{\mathbf{u}}}{dt}\right| = \frac{1}{(1+t^2)^{\frac{3}{2}}}\left\{\sqrt{1+t^2}\right\} = \frac{1}{1+t^2}$$

We can now find the unit normal vector

$$\hat{\mathbf{N}} = \frac{d\hat{\mathbf{u}}}{dt}\Bigg/\left|\frac{d\hat{\mathbf{u}}}{dt}\right| = \frac{1}{(1+t^2)^{\frac{3}{2}}}\left\{-t\,\mathbf{i} + \mathbf{j}\right\}(1+t^2)$$

example continued ...

$$\hat{\mathbf{N}} = \frac{1}{\sqrt{1+t^2}} \left\{ -t\,\mathbf{i} + 1\,\mathbf{j} \right\}$$

Finally we find the curvature κ

$$\kappa = \left| \frac{d\hat{\mathbf{u}}}{dt} \right| \Big/ |\mathbf{r}'| = \frac{1}{1+t^2} \frac{1}{\sqrt{1+t^2}}$$

$$\kappa = \frac{1}{(1+t^2)^{\frac{3}{2}}}$$