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THE NATIONAL UNIVERSITY OF IRELAND MAYNOOTH

MATHEMATICAL PHYSICS

EE112

Engineering Mathematics II

Vector Triple Products

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3 Vectors: Triple Products

3.1 The Scalar Triple Product

The scalar triple product, as its name may suggest, results in a scalar as its result. It is a means of combining three vectors via cross product and a dot product. Given the vectors

$$\begin{aligned}\mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}\\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}\\ \mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}\end{aligned}$$

a scalar triple product will involve a dot product and a cross product

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$$

It is necessary to perform the cross product before the dot product when computing a scalar triple product,

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

since $\mathbf{A} = A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}$ one can take the dot product to find that

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (A_1)\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} - (A_2)\begin{vmatrix} B_1 & B_3 \\ C_1 & C_3 \end{vmatrix} + (A_3)\begin{vmatrix} B_1 & B_2 \\ C_1 & C_2 \end{vmatrix}$$

which is simply

Important Formula 3.1.

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

The usefulness of being able to write the scalar triple product as a determinant is not only due to convenience in calculation but also due to the following property of determinants

Note 3.1. Exchanging any two adjacent rows in a determinant changes the sign of the original determinant.

Thus,

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = \begin{vmatrix} B_1 & B_2 & B_3 \\ A_1 & A_2 & A_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

Formula 3.1.

$$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C}) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}).$$

3.1.1 Worked examples.

Example 3.1.1. Given,

$$\begin{aligned} \mathbf{A} &= 2\,\mathbf{i} + 3\,\mathbf{j} - 1\,\mathbf{k} \\ \mathbf{B} &= -\,\mathbf{i} + \mathbf{j} \\ \mathbf{C} &= 2\,\mathbf{i} + 2\,\mathbf{j} \end{aligned}$$

Find

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

Solution:
Method 1:

Begin by finding

$$\begin{aligned} \mathbf{B}\times\mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ 2 & 2 & 0 \end{vmatrix} &= \mathbf{i}\begin{vmatrix} 1 & 0 \\ 2 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 0 \\ 2 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} \\ &= \big((1)(0) - (0)(2)\big)\mathbf{i} - \big((-1)(0) - (0)(2)\big)\mathbf{j} + \big((-1)(2) - (1)(2)\big)\mathbf{k} \\ &= 0\,\mathbf{i} + 0\,\mathbf{j} - 4\,\mathbf{k}. \end{aligned}$$

... example continued

Take the dot product with **A** to find

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= (2)(0)+(3)(0)+(-1)(-4)\\ &= 4\end{aligned}$$

Method 2:
Evaluate the determinant

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \begin{vmatrix} 2 & 3 & -1\\ -1 & 1 & 0\\ 2 & 2 & 0\end{vmatrix} = (2)\begin{vmatrix}1 & 0\\ 2 & 0\end{vmatrix} - (3)\begin{vmatrix}-1 & 0\\ 2 & 0\end{vmatrix} + (-1)\begin{vmatrix}-1 & 1\\ 2 & 2\end{vmatrix}\\ &= (2)\big((1)(0)-(0)(0)\big)-(3)\big((-1)(0)-(0)(2)\big)+(-1)\big((-1)(2)-(1)(2)\big)\\ &= 4\end{aligned}$$

Example 3.1.2. Prove that

Important Formula 3.2.

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

Solution:
Notice that there are no brackets given here as the only way to evaluate the scalar triple products is to perform the cross products before performing the dot products[a]. Let

$$\begin{aligned}\mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k}\\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k}\\ \mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k}\end{aligned}$$

now,

$$\mathbf{A}\cdot\mathbf{B}\times\mathbf{C} = \begin{vmatrix} A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\\ C_1 & C_2 & C_3\end{vmatrix} = -\begin{vmatrix} C_1 & C_2 & C_3\\ B_1 & B_2 & B_3\\ A_1 & A_2 & A_3\end{vmatrix} = \begin{vmatrix} C_1 & C_2 & C_3\\ A_1 & A_2 & A_3\\ B_1 & B_2 & B_3\end{vmatrix} = \mathbf{C}\cdot\mathbf{A}\times\mathbf{B} = \mathbf{A}\times\mathbf{B}\cdot\mathbf{C}$$

[a] This is due to the fact that if the dot product is evaluate first one would be left with a cross product between a scalar and a vector which is not defined.

3.2 The Vector Triple Product

The vector triple product, as its name suggests, produces a vector. It is the result of taking the cross product of one vector with the cross product of two other vectors.

Important Formula 3.3 (Vector Triple Product)**.**

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}$$

Proving the vector triple product formula can be done in a number of ways. The straightforward method is to assign

$$\begin{aligned} \mathbf{A} &= A_1\,\mathbf{i} + A_2\,\mathbf{j} + A_3\,\mathbf{k} \\ \mathbf{B} &= B_1\,\mathbf{i} + B_2\,\mathbf{j} + B_3\,\mathbf{k} \\ \mathbf{C} &= C_1\,\mathbf{i} + C_2\,\mathbf{j} + C_3\,\mathbf{k} \end{aligned}$$

and work out the various dot and cross products to show that the result is the same. Here we shall however go through a slightly more subtle but less calculation heavy proof.

Note 3.2. The vector $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ must be in the same plane as $\mathbf{B}$ and $\mathbf{C}$. This is due to fact that the vector that results from the cross product is perpendicular to both the vectors whose product has just been taken. Since one can say that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ is on the same plane as $\mathbf{B}$ and $\mathbf{C}$ it follows that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \alpha\mathbf{B} + \beta\mathbf{C}$$

where α and β are scalars.

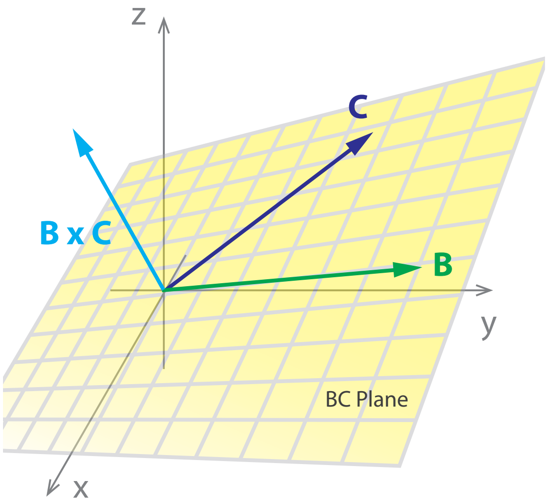
We introduce a new coordinate system with the unit vector $\mathbf{i}'$ along the vector $\mathbf{B}$, $\mathbf{j}'$ a unit vector (orthogonal to $\mathbf{i}'$), which is on the same plane as the both the vectors $\mathbf{B}$ and $\mathbf{C}$, and $\mathbf{k}'$ a unit vector orthogonal to both $\mathbf{i}'$ and $\mathbf{j}'$[1]. Using this *basis* allows one to write the vectors $\mathbf{B}$ and $\mathbf{C}$ as

$$\begin{aligned} \mathbf{B} &= B_1\mathbf{i}' + 0\mathbf{j}' + 0\mathbf{k}' \\ \mathbf{C} &= C_1\mathbf{i}' + C_2\mathbf{j}' + 0\mathbf{k}' \end{aligned}$$

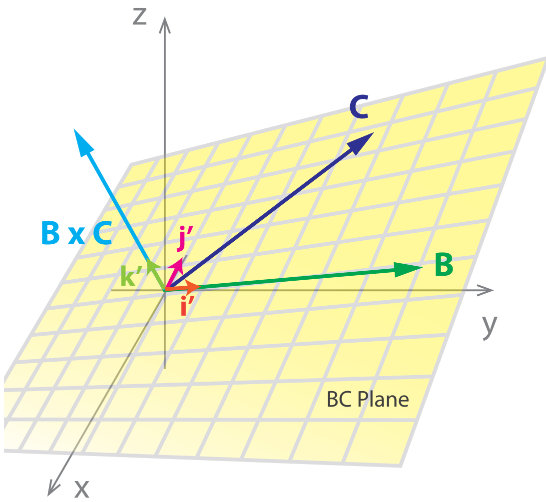
however there is no special reduction to the representation of the vector $\mathbf{A}$ in terms of this new basis thus,

$$\mathbf{A} = A_1\mathbf{i}' + A_2\mathbf{j}' + A_3\mathbf{k}'.$$

[1] the unit vector $\mathbf{k}'$ will thus point in the same direction as the vector $\mathbf{B} \times \mathbf{C}$.



(a) The vectors **B** and **C** define the BC-plane.



(b) The unit vectors $\mathbf{i}'$ and $\mathbf{j}'$ are on the BC-plane while $\mathbf{k}'$ points in the same direction as $(\mathbf{B} \times \mathbf{C})$

Figure 1: Figures representing the change in basis.

We know that the vector $\mathbf{B} \times \mathbf{C}$ must be of the form of $0\mathbf{i}' + 0\mathbf{j}' + \gamma\mathbf{k}'$ for some scalar γ. We find the value of γ by taking the cross product

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ B_1 & 0 & 0 \\ C_1 & C_2 & 0 \end{vmatrix} = \mathbf{i}' \begin{vmatrix} 0 & 0 \\ C_2 & 0 \end{vmatrix} - \mathbf{j}' \begin{vmatrix} B_1 & 0 \\ C_1 & 0 \end{vmatrix} + \mathbf{k}' \begin{vmatrix} B_1 & 0 \\ C_1 & C_2 \end{vmatrix} = 0\mathbf{i}' + 0\mathbf{j}' + B_1 C_2 \mathbf{k}'$$

We have now found that

$$\mathbf{B} \times \mathbf{C} = B_1 C_2 \mathbf{k}'$$

.

Now examining the final cross product

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} \mathbf{i}' & \mathbf{j}' & \mathbf{k}' \\ A_1 & A_2 & A_3 \\ 0 & 0 & B_1 C_2 \end{vmatrix} = A_2 B_1 C_2 \mathbf{i}' - A_1 B_1 C_2 \mathbf{j}' + 0\mathbf{k}'.$$

thus,

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = A_2 B_1 C_2 \mathbf{i}' - A_1 B_1 C_2 \mathbf{j}'$$

.

Here a clever addition of zero is useful

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= A_2 B_1 C_2 \mathbf{i}' - A_1 B_1 C_2 \mathbf{j}' \underbrace{+ A_1 B_1 C_1 \mathbf{i}' - A_1 B_1 C_1 \mathbf{i}'}_{0} \\ &= (A_2 C_2 + A_1 C_1) B_1 \mathbf{i}' - A_1 B_1 (C_1 \mathbf{i}' + C_2 \mathbf{j}'). \end{aligned}$$

This is the desired result as returning to our definitions of **A**, **B**and **C** in this basis,

$$\begin{aligned}
\mathbf{A} &= A_1\mathbf{i}' + A_2\mathbf{j}' + A_3\mathbf{k}' \\
\mathbf{B} &= B_1\mathbf{i}' \\
\mathbf{C} &= C_1\mathbf{i}' + C_2\mathbf{j}'
\end{aligned}$$

one finds that,

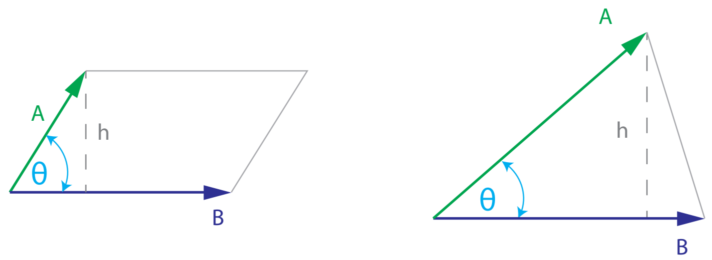
$$\begin{aligned}
\mathbf{A}\cdot\mathbf{B} &= A_1B_1 \\
\mathbf{A}\cdot\mathbf{C} &= A_1C_1 + A_2C_2
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= (A_2C_2 + A_1C_1)B_1\mathbf{i}' - A_1B_1(C_1\mathbf{i}' + C_2\mathbf{j}') \\
&= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}.
\end{aligned}$$

3.3 Area and Volume Using a Cross Product

3.3.1 The area of a parallelogram.



(a) The parallelogram (b) The triangle

Figure 2: Areas related to the cross product.

The are of a parallelogram is simply given by the product of the base and the height of the parallelogram. Here this is given by

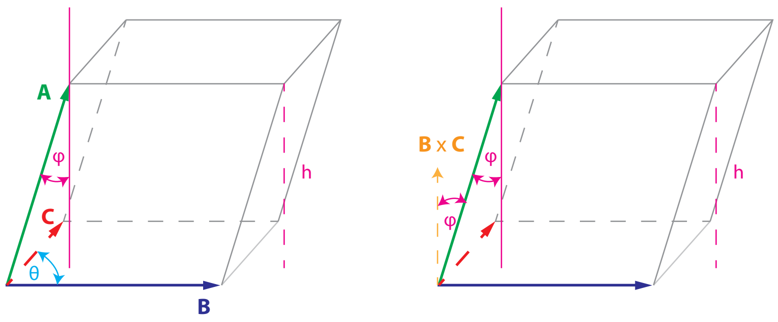
$$\begin{aligned}\text{Area of parallelogram} &= h\,|\,\mathbf{B}\,| \\ &= (|\,\mathbf{A}\,|\sin(\theta))\,|\,\mathbf{B}\,| \\ &= |\,\mathbf{A}\,|\,|\,\mathbf{B}\,|\sin(\theta) \\ &= |\mathbf{A}\times\mathbf{B}|\end{aligned}$$

3.3.2 The area of a triangle.

The area of a triangle is half the base times the height. From the figure we have

$$\begin{aligned}\text{Area of triangle} &= \frac{1}{2}\,|\,\mathbf{B}\,|\,h = \frac{1}{2}\,|\,\mathbf{B}\,|\,(|\,\mathbf{A}\,|\sin(\theta)) \\ &= \frac{1}{2}\,|\mathbf{A}\times\mathbf{B}|\end{aligned}$$

3.3.3 The volume of a parallelepiped.



(a) The angle between **B** and **C** is θ.

(b) The angle between h and **A** is ϕ.

Figure 3: A parallelepiped.

ADVANCED ASIDE 3.1. A parallelepiped is a three dimensional object whose six sides are parallelograms. The volume of a parallelepiped is given by

$$V = (\text{Base Area})(\text{Height})$$

The area of the base is the area of a parallelogram as such one has

$$\text{Area of the Base} = |\mathbf{B} \times \mathbf{C}|$$

The height h requires a little geometry but is simply

$$h = |\,\mathbf{A}\,| \cos(\phi)$$

notice that the vector $\mathbf{B} \times \mathbf{A}$ is parallel to the line h. Thus the vector $\mathbf{A}$ makes an angle[a] with the vector $\mathbf{B} \times \mathbf{C}$ of $\gamma = \phi$. Finally we have the volume of the parallelepiped given by

$$\begin{aligned}
\text{Volume of parallelepiped} &= (\text{Base})(\text{height}) \\
&= (|\mathbf{B} \times \mathbf{C}|)\,(|\,\mathbf{A}\,|\,|\cos(\gamma)|) \\
&= |\,\mathbf{A}\,|\,|\mathbf{B} \times \mathbf{C}|\,|\cos(\gamma)| \\
&= |\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})|
\end{aligned}$$

[a] It is also possible for $\mathbf{B} \times \mathbf{C}$ to make an angle $\gamma = 180° - \phi$ which does not affect the result since $|\cos(180° - \phi)| = |\cos(\phi)|$

3.4 Summary of Vector Rules

Here we list most of the main results concerning vectors,

$$\mathbf{A}\cdot\mathbf{A} = A^2 \equiv |\,\mathbf{A}\,|^2 \qquad (3.4.1) \qquad\qquad \mathbf{A}\times\mathbf{A} = \mathbf{0} \qquad (3.4.5)$$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \qquad (3.4.2) \qquad\qquad \mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A} \qquad (3.4.6)$$

$$\mathbf{A}\cdot(\alpha\mathbf{B}) = \alpha(\mathbf{A}\cdot\mathbf{B}) \qquad (3.4.3) \qquad\qquad \mathbf{A}\times(\alpha\mathbf{B}) = \alpha(\mathbf{A}\times\mathbf{B}) \qquad (3.4.7)$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\times\mathbf{B})\cdot\mathbf{C} \qquad (3.4.4) \qquad\qquad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} \qquad (3.4.8)$$

3.4.1 Worked problem

Example 3.4.1 (Manipulating vectors without evaluation)**.**
Prove that

(i) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})$

(ii) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D})$

Solution:

(i) let $\mathbf{U} = \mathbf{A}\times\mathbf{B}$

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= \mathbf{U}\times(\mathbf{C}\times\mathbf{D}) \\
&= (\mathbf{U}\cdot\mathbf{D})\mathbf{C} - (\mathbf{U}\cdot\mathbf{C})\mathbf{D} && \text{using Eq 3.4.8} \\
&= (\mathbf{A}\times\mathbf{B}\cdot\mathbf{D})\mathbf{C} - (\mathbf{A}\times\mathbf{B}\cdot\mathbf{C})\mathbf{D} \\
&= (\mathbf{A}\cdot\mathbf{B}\times\mathbf{D})\mathbf{C} - (\mathbf{A}\cdot\mathbf{B}\times\mathbf{C})\mathbf{D} && \text{using Eq 3.4.4} \\
&= \mathbf{C}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{D}) - \mathbf{D}(\mathbf{A}\cdot\mathbf{B}\times\mathbf{C}).
\end{aligned}$$

(ii) let $\mathbf{V} = (\mathbf{C}\times\mathbf{D})$

$$\begin{aligned}
(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) &= -(\mathbf{C}\times\mathbf{D})\times(\mathbf{A}\times\mathbf{B}) && \text{using Eq 3.4.6} \\
&= -\mathbf{V}\times(\mathbf{A}\times\mathbf{B}) \\
&= -\{(\mathbf{V}\cdot\mathbf{B})\mathbf{A} - (\mathbf{V}\cdot\mathbf{A})\mathbf{B}\} && \text{using Eq 3.4.8} \\
&= (\mathbf{C}\times\mathbf{D}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\times\mathbf{D}\cdot\mathbf{B})\mathbf{A} \\
&= (\mathbf{C}\cdot\mathbf{D}\times\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{D}\times\mathbf{B})\mathbf{A} && \text{using Eq 3.4.4} \\
&= \mathbf{B}(\mathbf{A}\cdot\mathbf{C}\times\mathbf{D}) - \mathbf{A}(\mathbf{B}\cdot\mathbf{C}\times\mathbf{D}).
\end{aligned}$$