# Introduction to the Laplace Transform

## Some comments

- Finding the solution to a differential equation is not easy.
- Are there any other ways to do this?
- Yes there are? It would be nice if we could transform differential equations into algebraic equations, solve these, and then transform back again to get the solution to differential equations,
- The Laplace transform provides one such method of doing this.

## The Laplace transfrom

Let f(t) be a given function defined for all t greater than or equal to zero. Then, provide the following integral exists, we define the Laplace transform of f(t) to be:

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

The operation that yields F(s) from f(t) is called the **Laplace Transform.** F(s) is denoted L(f).

Furthermore, the original function f(t) is called the inverse transform of F(s) and is denoted:

$$f(t) = L^{-1}(F)$$

# Linearity of the Laplace transform

From the definition of the Laplace transform we can show:

$$L(f+g)=L(f)+L(g)$$

Where f(t) and g(t) are functions.

**Complete this is class from the definition.**

Because of this property, the Laplace transform is said to be a linear operator.

# Laplace transforms of elementary functions

Find the Laplace transforms of the following functions:

$(i) \quad f(t) = 1;$

$(ii) \quad f(t) = k;$

$(iii) \quad f(t) = e^{at};$

$(iv) \quad f(t) = \cos(at);$

$(v) \quad f(t) = t;$

$(vi) \quad f(t) = t^{n};$

$(vii) \quad f(t) = \cosh(at).$

where all functions are defined for all t greater than zero.

## Laplace transforms of elementary functions

Find the Laplace transforms of:

$$(i)\quad f(t) = 1 + 2t + 3t^2;$$
$$(ii)\quad f(t) = 5e^{2t} - 3e^{-t}.$$

where all functions are defined for all t greater than zero.

## Laplace transforms of elementary functions

Find the Laplace transforms of:

$$(i) \quad f(t) = 6\sin(3t) - 4\cos(5t);$$
$$(ii) \quad f(t) = 2\cosh(2t) - \sinh(3t).$$

where all functions are defined for all t greater than zero.

## Laplace transforms of elementary functions

Find the Laplace transforms of:

$$(i)\quad f(t) = \sin(at)$$
$$(ii)\quad f(t) = \cosh(at)$$
$$(iii)\quad f(t) = t^2$$

where all functions are defined for all t greater than zero.

## Laplace transforms of elementary functions

Find the Laplace transforms of:

$$(i) \quad f(t) = \sin^2(t)$$

$$(ii) \quad f(t) = \sin(t^2)$$

$$(iii) \quad f(t) = 3\sin(at + b)$$

where all functions are defined for all t greater than zero.

# Properties of the Laplace Transform

## The Laplace transfrom

The Laplace transform is very useful because of its properties.

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt$$

and because of the way these properties can be used to make solving differential equations simple.

We shall see that making use of these properties to manipulate differential equations will be very useful.

## 1. Linearity

**The Laplace transform is a linear operation.**

$$L(f(t)+g(t)) = F(s)+G(s)$$

**Proof:**

## 2. Frequency shifting

**Multiplication by an exponential is a shift in the s-domain.**

$$L(e^{at} f(t)) = F(s-a)$$

**Proof.**

# Examples

Find the Laplace transforms of:

$$(i) \quad f(t) = e^{bt}\sin(at)$$
$$(ii) \quad f(t) = \sin(at)$$
$$(iii) \quad f(t) = e^{at}$$

where all functions are defined for all t greater than zero.

## Differentiation

**The Laplace transform of a derivative:**

$$L(\frac{df}{dt}) = sL(s) - f(0)$$

**Proof**

## Differentiation

**The Laplace transform of a derivative:**

$$L\left(\frac{d^2 f}{dt^2}\right) = s^2 L(s) - sf(0) - \left.\frac{df}{dt}\right|_{t=0}$$

**Proof**

## Examples

Find the Laplace transforms of:

$$(i) \quad f(t) = k$$
$$(ii) \quad f(t) = t$$
$$(iii) \quad f(t) = t^n$$

where all functions are defined for all t greater than zero.

# Inverse Laplace Transform

## Some comments

- If the Laplace transform of a function f(t) is F(s), the f(t) is called the inverse transform of F(s) and is written:

$$f(t) = L^{-1}(F(s))$$

- For example: since

$$L(1) = \frac{1}{s} \Rightarrow L^{-1}(\frac{1}{s}) = 1.$$

- Similarly:

$$L(\cos(t)) = \frac{s}{s^2 + 1} \Rightarrow L^{-1}(\frac{s}{s^2 + 1}) = \cos t.$$

## The inverse Laplace transfrom

Usually engineers use tables to find inverse Laplace transforms. Of course, in this set of lectures we will not do this.

**Example:** Find the inverse Laplace transforms of the following:

$$(i)\quad \frac{1}{s^2+9}$$

$$(ii)\quad \frac{5}{2s+9}$$

$$(iii)\quad \frac{6}{s^3}$$

$$(iv)\quad \frac{3}{(s+2)^2}$$

## The inverse Laplace transfrom

The key to solving differential equations is being able to find inverse transforms. Most of the Laplace transforms that we looked at so far were of the form

$$\frac{F(s)}{G(s)}$$

Where F(s) and G(s) are polynomials in s. One nice method to Invert transforms is to use the method of partial fractions. This method assumes that F(s) and G(s) have real coefficients, no common factors and that the degree of F(s) is lower than that of G(s).

## The inverse Laplace transfrom

Where F(s) and G(s) are polynomials in s. One nice method to Invert transforms is to use the method of partial fractions. This method assumes that F(s) and G(s) have real coefficients, no common factors and that the degree of F(s) is lower than that of G(s).

$$\frac{F(s)}{G(s)}$$

To use this method we recall the inverse transform of several simple functions.

## Some more examples

.

**Example:** Find the inverse Laplace transforms of the following:

$$(i)\quad \frac{a}{s^2 + a^2}$$

$$(ii)\quad \frac{5}{s + a}$$

$$(iii)\quad \frac{b}{s^3}$$

$$(iv)\quad \frac{2}{(s+3)^2 + 1}$$

## Inverse Laplace transforms using partial fractions

Sometimes the function whose inverse is required is not in the standard type. In such cases we can often use partial fractions to find the invese transform. Partial fractions is based on the observation that rational transfer functions can often be written in terms of their partial fractions.

$$Y(s) = \frac{F(s)}{G_1(s)G_2(s)\cdots G_n(s)} = \frac{A_1(s)}{G_1(s)} + \cdots + \frac{A_n(s)}{G_n(s)}$$

**Example:**

$$Y(s) = \frac{2s - 3}{s(s - 3)} = \frac{1}{s} + \frac{1}{s - 3}$$

## Inverse Laplace transforms using partial fractions

The basic idea with this method is then to factor the denomenator into elementary functions, whose inverse transform is known to us. Some common factors of practical interest are:

$$\text{Case 1 (unrepeated factors)}: s - a$$
$$\text{Case 2 (repeated factors)} \quad : (s - a)^m$$
$$\text{Case 3 (complex factors)} \quad : (s - a)(s - a^*)$$
$$\text{Case 4 (rep. complex factors)}: \left((s - a)(s - a^*)\right)^m$$

## Some more examples

**Example:** Find the inverse Laplace transforms of the following:

$$(i) \quad \frac{1}{(s-1)(s^2+1)}$$

$$(ii) \quad \frac{5}{s(s+1)}$$

$$(iii) \quad \frac{1}{s(s-17)^3}$$

$$(iv) \quad \frac{2s^2+1}{s((s+3)^2+1)}$$

## A very brief comment: Poles and zeros

Engineers use Laplace transforms to solve lots of problems. In fact, engineers see the world through the eyes of the Laplace transform, and interpret properties of systems in terms of the pole and zeros of a transfer function. Recall, transfer functions have the general form:

$$\frac{F(s)}{G(s)}$$

Sometimes we call this function a transfer function. Roughly speaking, the values of s for which G(s)=0, are called the **poles of the transfer function**; the values of s for which F(s)= 0 are called the **zeros of the transfer function**.

# Solving differential equations using the Laplace transform

## The inverse Laplace transform

Much of the interest in the Laplace transform is that it provides a method that can be used to solve linear differential equations. Roughly, this method can be summarised as follows.

1. Take the Laplace transform of the left and right hand side of the differential equations.

2. Write Y(s)=H(s)F(s), taking care to include initial conditions.

3. Determine y(t) using inverse transforms.

## The inverse Laplace transform

**Example:** Solve the differential equation:

$$2\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = 0;$$

Given that when t = 0, y = 4 and the derivative of y with respect to t is 9.

## The inverse Laplace transform

**Example:** Solve the differential equation:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 13y = 0;$$

Given that when t = 0, y = 3 and the derivative of y with respect to t is 7.

## The inverse Laplace transform

**Example:** Solve the differential equation:

$$\frac{d^2y}{dt^2} - 7\frac{dy}{dt} + 10y = 20 + e^t;$$

Given that when t = 0, y = 0 and the derivative of y with respect to t is 0.