Basic concepts

- An ordinary differential equation is an equation that contains one or several derivatives of an unknown function.

- Examples:

$$(i) \quad \frac{dy}{dt} = 1$$

$$(ii) \quad \frac{dy}{dt} = \cos(t)$$

$$(iii) \quad t^2 \frac{dy}{dt} = (t + y)^{23}$$

- Differential equations are extremely important because we use them to model the world.

Basic concepts

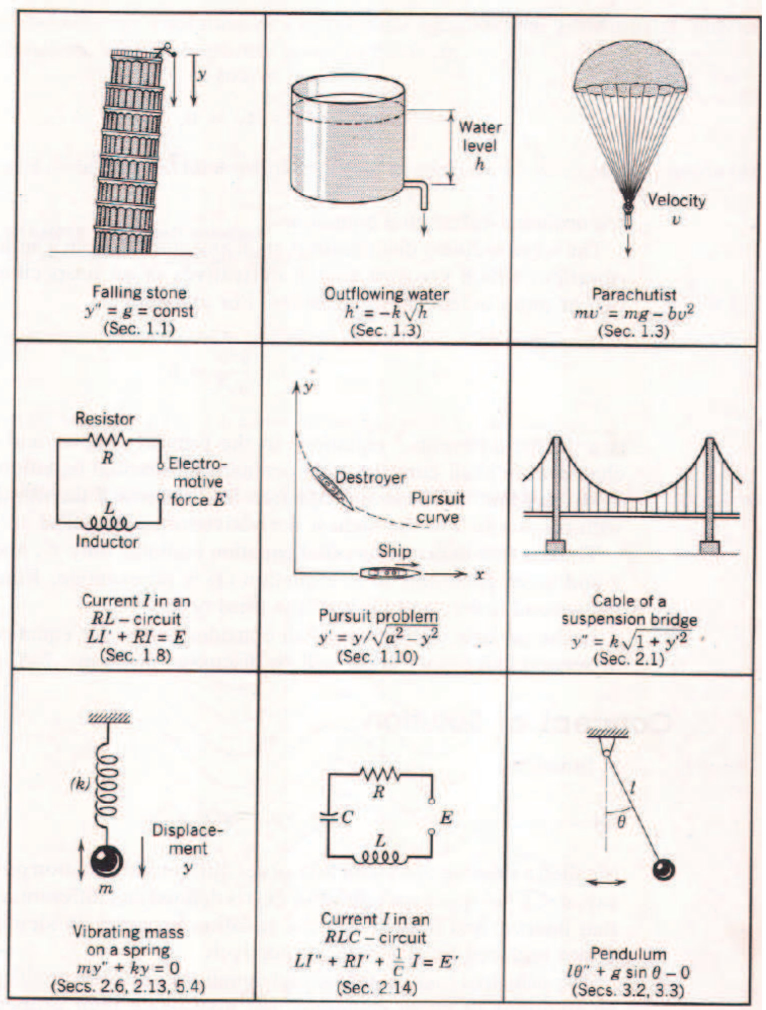

Fig. 1. Some applications of differential equations

Examples

(1) For example: A stone falling from a height h.

$$\frac{dy}{dt} = g$$

(1) The current in a LCR circuit.

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = \frac{d^2V}{dt^2}$$

Solutions

A solution to a given differential equation on some open interval $a < t < b$ is a function $y = f(t)$ that satisfies the differential equation on that interval.

Example:

$$(i) \quad t\frac{dy}{dt} = 2y; \qquad solution: y = t^2$$

There are many issues related to the existence of solution and the uniqueness of solutions, but these are for another course.

A basic requirement for an engineer is to be able to solve differential equations.

Solving differential equations

Most of the time we solve differential equations numerically.
Sometimes when we are lucky we can solve them analytically.

Example: Solve the differential equation:

$$9y\frac{dy}{dt} + 4t = 0 \Rightarrow 9ydy = -4tdt$$
$$\Rightarrow \frac{9y^2}{2} + 2t^2 + C = 0.$$

Example: Solve the differential equation:

$$\frac{dy}{dt} + y = 0 \Rightarrow \frac{dy}{y} = dt$$
$$\Rightarrow \ln y = t + C$$

Some comments

- Up to now we have dealt with first order linear differential equations.
- First order refers to the fact that the number of times the unknown function is differentiated is 1.
- The equations are linear because the equations are linear in the unknown variable y.
- Most real world systems are not first order and are not linear.
- When they are linear they can be solved easily.

Second order differential equations

A second order differential equation is an equation of the form:

$$\frac{d^2y}{dt^2} + f(t)\frac{dy}{dt} + g(t)y = r(t)$$

The equation is nonlinear if it cannot be written in this form.

If r(t)=0; the equation is called a homogeneous equation.

Otherwise the equation is said to be non-homogeneous.

First theorem on differential equations

Theorem: For a homogeneous linear differential equation, any linear combination of two solutions on an open interval I, is again a solution on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.

Proof:

Initial value problems

We have already seen that solutions to differential equations involve unknown constants.

$$\frac{dy}{dt} + y = 0 \Rightarrow \frac{dy}{y} = dt$$
$$\Rightarrow \ln y = -t + C$$

To get rid of these constants we need some more informtion about the solution; for example some of the initial conditions.

Initial value problems

Example: Solve the initial value problem

$$\frac{d^2y}{dt^2} - y = 0; y(0) = 4; \frac{dy}{dt} = -2$$

The form of the equations suggests the solution. It says that

$$\frac{d^2y}{dt^2} = y$$

Clearly the matrix exponential is one such function. So let us try

as a solution

$$y = C_1e^t + C_2e^{-t}$$

Second order differential equations

Suppose that we have a differential equation of the form:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

There are many engineering situations described by this equation. You should be able to verify that the charge decaying across a capacitance in a LCR circuit satisfies this equation.

How do we solve such an equation?

We have already seem that matrix exponentials play a role in the solution of differential equations.

Ans:

Second order differential equations

Lets us guess that the solution to:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

is of the form:

$$y(t) = Ae^{mt}.$$

Substitution reveals that for this to be true:

$$aAm^2e^{mt} + bAme^{mt} + cAe^{mt} = 0;$$
$$Ae^{mt}(am^2 + bm + c) = 0.$$

Second order differential equations

Thus we have a solution to the equation:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

provided that:

$$Ae^{mt}(am^2 + bm + c) = 0.$$

for all t, or in otherwords:

$$(am^2 + bm + c) = 0.$$

Ans:

Second order differential equations

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Ans:

Second order differential equations

The equation

$$\boxed{(am^2 + bm + c) = 0.}$$

is called the **Auxilliary Equation**. Since the equation is quadratic in m, 3 types of roots are possible.

1. Two distinct real roots (when);
2. Two identical real roots (when);
3. Complex roots (when).

The different types of roots are determined by the values of the constants (a,b,c).

Second order differential equations

Types of solutions $\boxed{(am^2 + bm + c) = 0.}$

1. Roots of the Auxiliary Equation are real and different.

$$m = \alpha, m = \beta \Rightarrow y(t) = Ae^{\alpha t} + Be^{\beta t}$$

2. Roots of the Auxiliary Equation are the same.

$$m = \alpha \Rightarrow y(t) = Ate^{\alpha t} + Be^{\alpha t}$$

3. Roots of the Auxiliary Equation are complex.

$$m = \alpha \pm j\beta \Rightarrow y(t) = e^{\alpha t}\left(A\cos(\beta t) + B\sin(\beta t)\right)$$

These are general solutions. Particular solutions are obtained from initial values.

Example

Example: Determine the general solution of the

$$2\frac{d^2y}{dt^2}+5\frac{dy}{dt}-3y=0.$$

Find also the particular solution given that when t=0, y = 4, and that the first derivative of y with respect to t is 9.

Example

Example: Determine the general solution of the

$$9\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 16y = 0.$$

Find also the particular solution given that when t=0, y = 3, and that the first derivative of y with respect to t is 3.

Example

Example: The Equation:

$$\frac{d^2 i}{dt^2} + (R/L)\frac{di}{dt} + (1/LC)i = 0.$$

represents a current i flowing in an electrical circuit containing resistance R, inductance L, and capacatence C, connected in series. If R = 200 ohms; L = 0.20 henry; and C = 20e-6 Farads, solve the equation for i given the boundary conditions that when t = 0, i = 0, and

$$\frac{di}{dt} = 100.$$

Second order differential equations

Suppose now that we have a differential equation of the form:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t).$$

We usually refer to f(t) as an input or driving term. How do we solve such an equation?

In this case, the general solution will contain two terms. The first terms is any solution of ($y_1(t)$):

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0.$$

and the second is any solution of ($y_2(t)$)

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t).$$

Second order differential equations

The function $y_1(t)$ is called the complementary function (CF) and the function $y_2(t)$ is called the particular integral.

As we have seen, finding the complementary function is easy. Finding the particular integral is a little harder. But assuming we can find these, the boundary conditions are used to find any arbitrary constants floating around.

Example

Example: Determine the solution of the

$$9\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 16y = 4.$$

Find also the particular solution given that when t=0, y = 3, and that the first derivative of y with respect to t is 3.

Example

Example: Determine the general solution of the

$$9\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 16y = t + 1.$$

Find also the particular solution given that when t=0, y = 3, and that the first derivative of y with respect to t is 3.

Example

Example: Determine the general solution of the

$$9\frac{d^2y}{dt^2} - 24\frac{dy}{dt} + 16y = \cos t.$$

Find also the particular solution given that when t=0, y = 3, and that the first derivative of y with respect to t is 3.